Large Deviations Studies for Small Noise Limits of Dynamical Systems Perturbed by Lévy Processes

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"It is impossible to be a mathematician without being a poet in soul."
Sofya Kovalevskaya

À minha mãe e ao meu avô.

"Humility and perseverance are essential in the study of any science. In mathematics they are character builders."

E. Frenkel

Abstract

The first problem that we address in this thesis is the first exit from a fixed bounded domain for a certain class of exponentially light jump diffusions. We consider a gradient flow associated with a deterministic differential equation and perturbed, in small noise intensity ε, by a compensated compound Poisson process whose intensity is the product measure of the Lebesgue measure in a fixed time interval with a measure exponentially light with respect to the Lebesgue measure in the space of jumps, of the form $\nu(dz) = e^{-|z|^\alpha} dz$ for some $\alpha > 0$, rescaled by $\frac{1}{\varepsilon}$, the inverse of the noise parameter. We are interested in understanding asymptotically, when the parameter that tunes the noise vanishes ($\varepsilon \to 0$), the law and the expected value of the first exit time from a certain pre-fixed domain containing the stable state of the underlying dynamical system. When $\alpha \geq 1$, in the superexponential regime, it is deduced that the exit occurs on a large deviations scale in terms of a Poissonian rate function. When $\alpha < 1$, in the subexponential regime, we study the first exit time of the renormalized deviation process from the unperturbed dynamical system, which is achieved with a moderate deviations speed ε^α according to a certain rate function that has a quadratic form. Nevertheless, in both regimes, the higher the rate function the less efficient the path is and the exit depends on the minimal energy that the jump diffusion needs to spend in order to follow a possible controlled path that leads to the exit.

The second problem studied in this work is the small noise limit of a coupled forward-backward system of stochastic differential equations (FBSDEs for short). The class of FBSDEs that we study contains the paradigmatic nonlocal toy-model known as the fractal Burgers equation, which is a mathematical idealization for the velocity of a compressible fluid flow affected by external and internal non-local forces. We prove in a suitable functional space the convergence of the FBSDE system when both sources of noise, one Brownian and another Poissonian, converge to zero to the deterministic limiting differential equations. We study via this probabilistic approach the convergence at the level of viscosity solutions of the partial-integral differential equation associated to the FBSDE system. Our last result is a large deviations statement for the laws of the forward and backward processes. The key to derive the large deviations principle is the representation of the backward process, via a deterministic function, in terms of the forward process which decouples the two equations.

Keywords: Large deviations principle, weak convergence, first exit times, Lévy processes, Poisson random measures, exponentially light jump diffusions, forward-backward stochastic differential equations, viscosity solutions of partial-integral differential equations.

Zusammenfassung

Der erste Fragekomplex in dieser Arbeit behandelt das Problem der (ersten) Austrittszeiten aus einem beschränkten Gebiet für eine bestimmte Klasse von *exponentiell leichten* Sprungdiffusionen. Betrachtet wird der Gradientenfluss zu einer deterministischen Differenzialgleichung mit einem Rauschterm von kleiner Intensität ε. Dieser sei gegeben durch einen zusammengesetzten Poisson-Prozess (*compound Poisson process*) dessen Intensität das mit dem inversen Rauschlevel $\frac{1}{\varepsilon}$ reskalierte Produktmaß aus dem Lebesgue-Maß des Zeitintervalls und einem exponentiell leichtem Maß bezüglich des Lebesgue-Maßes im Sprungraum ist. Dieses ist von der Form $\nu(dz) = e^{-|z|^\alpha} dz$ für ein $\alpha > 0$.

Unser Interesse gilt der Asymptotik von Verteilung und Erwartung der Austrittszeit aus einer Umgebung des stabilen Punktes des zugrundeliegenden dynamischen Systems für verschwindendes Rauschlevel ($\varepsilon \to 0$).

Im superexponentiellen Regime, $\alpha \geq 1$, folgt der Austritt einem Prinzip der großen Abweichungen mit Poisson'scher Ratenfunktion. Im subexponentiellen Regime, $\alpha < 1$ erfolgt der Austritt über moderate Abweichungen mit der Rate ε^α und einer quadratischen Ratenfunktion.

In beiden Fällen wird die Austrittszeit von der minimalen Energie bestimmt, die die Sprungdiffusion aufwenden muss um einem zum Austritt führenden Kontrollpfad zu folgen.

Der zweite Fragekomplex betrachtet den Limes für kleines Rauschen eines gekoppelten vorwärts-rückwärts Systems stochastischer Differentialgleichungen (kurz FBSDE). Die betrachtete Klasse von FBSDEs enthält die fraktionale Burgers-Gleichung, eine mathematische Idealisierung der Geschwindigkeit eines kompressiblen Fluidflusses unter Einfluss externer und interner nicht-lokaler Kräfte.

Wir beweisen die Konvergenz des FBSDE Systems in geeigneten Funktionalräumen gegen eine deterministische Limit-Differenzialgleichung wenn sowohl eine Brown'sche als auch eine Poisson'sche Rauschkomponente verschwinden.

Unser letztes Resultat ist ein Prinzip der großen Abweichungen für die Verteilungen des Vorwärts- und des Rückwärtsprozesses. Der Schlüssel dazu ist die Darstellung des Rückwärtsprozesses als eine deterministische Funktion des Vorwärtsprozesses sodass die zwei Gleichungeng sich entkoppeln.

Schlagwörter: Prinzip der großen Abweichungen, schwache Konvergenz, erste Austrittszeiten, Lévy Prozesse, Poisson'sche Zufallsmaße, exponentiell leichte Sprungdiffusionen, vorwärts-rückwärts stochastische Differentialgleichungen, Viskositätslösungen partieller Integro-Differentialgleichungen.

Contents

Introduction

Roughly speaking, the *Law of Large Numbers* is a statement about the convergence in probability of the sample average of a family of i.i.d. random variables with expected value μ and variance σ^2 to the expected value μ. The *Central Limit Theorem* asserts the convergence in distribution of the renormalized deviation from the expected value of the arithmetic mean of a family of i.i.d. random variables with first two moments to a Gaussian law. A large deviations statement is a much finer asymptotic analysis that concerns with the exponential decay of probabilities of unlikely events with respect to an associated parameter in terms of a certain functional.
Fixed a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a topological space $\mathcal{S}$, if $A \in \mathcal{B}(\mathcal{S})$ is a Borel subset of $\mathcal{S}$, a *large deviations principle* for a family of $\mathcal{S}$-valued random variables $(X^\varepsilon)_{\varepsilon>0}$ is concerned with the evaluation of the probability

$$\mathbb{P}(X^\varepsilon \in A) \simeq_\varepsilon e^{-\frac{1}{\varepsilon}I(A)} \quad \text{as } \varepsilon \to 0,$$

and the study of the rate of exponential decay $A \mapsto I(A)$ in terms of a functional $I : \mathcal{S} \longrightarrow [0, \infty]$ called rate function. In some classical references such as *Freidlin and Wentzell (1988)*, more linked to physical applications, I is called action functional. In the expression above, the study of the exponential decay of that probability is stated for the order of convergence ε^{-1} as $\varepsilon \to 0$. But it is natural to ask about the study of the probabilities of such unlikely events with exponential decay but with a lower order of convergence than ε^{-1} as $\varepsilon \to 0$. This is the core of a *moderate deviations principle.* A moderate deviations principle corresponds to the study of the asymptotics of

$$\frac{\varepsilon}{a^2(\varepsilon)} \ln \mathbb{P}(X^\varepsilon \in .),$$

with $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ a measurable function such that $a(\varepsilon) \to 0$ and $\frac{\varepsilon}{a^2(\varepsilon)} \to 0$ as $\varepsilon \to 0$. In this sense a moderate deviations principle bridges the gap between the central limit approximation and a large deviations statement.

Large deviations theory is a very fruitful and mature field nowadays and one of the most popular interface areas of probability with other branches of mathematics, such as convex analysis, functional analysis, partial differential equations and others. Historically, large deviations theory (LDT) made its first appearance in 1877 (see *Boltzmann (1877)*) in the context of Boltzmann's studies of the second law of thermodynamics. In the actual terminology, Boltzmann's discovery was how to express the asymptotic behaviour of certain multinomial probabilities in terms of the relative entropy of the system. But rapidly this branch of mathematics evolved with a diversity of applications in other fields, especially after the landmark work *Varadhan (1966)*. As classical references we refer the reader to *Dembo and Zeitoni (1998)*, *Deuschel and Strook (1989)*, *Ellis (1999)*, *Hollander (2000)*, *Ramasubramanian (2008)*, *Varadhan (1984)* and *Varadhan (2008)*. With physical examples in mind, large deviations principles can refer to extreme events such as a system that exchanges from one equilibrium state to another and that occur with a small probability. We refer to *Ellis (1985)* and *Freidlin-Wentzell (1988)* where applications of large deviations to statistical mechanics and to the study of metastable systems are respectively developed .

In this thesis we import several techniques from large/moderate deviations theory for Poisson random measures in order to understand asymptotically, as the source of noise vanishes, qualitative features concerning the first exit time problem for a certain class of jump diffusions and the small noise limit of a forward backward system of stochastic differential equations(FBSDE for short) with jumps.
One of the major difficulties of establishing large deviations principles for Poisson random measures is the non-existence of an analogous Cameron-Martin theorem in the Poissonian space and the highly nonlinear structure encoded in the Poisson random measures that is inherited from the jumps of the underlying Lévy processes. Several breakthrough works were made in the direction of establishing large deviations for Lévy processes and the underlying Poisson random measures. We cite the works *A. de Acosta (1994)*, *A. de Acosta (1997)*, *Borovkov (1967)*, *Florens and Pham (1998)* and *Leonard (2000)*, which state large deviations principles for Poisson random measures (PRMs for short), under different exponential integrability conditions for the PRM that are considered.

It was proved in *Varadhan (1966)*, under some suitable assumptions on the topological space $\mathcal{S}$ where the family $(X^\varepsilon)_{\varepsilon>0}$ takes values, the equivalence between the large deviations principle and a variational principle, called later *Laplace-Varadhan principle*. Typically the strategy to verify the *Laplace-Varadhan principle* is to reduce it to the verification of simpler variational formulas through the use of the concept of *relative entropy* with the help of the *Donsker-Varadhan theorem*. This is the so-called *weak convergence approach to large deviations* that relies on the use of arguments from weak convergence of probability measures, respectively the laws of the family $(X^\varepsilon)_{\varepsilon>0}$ that obeys a large deviations principle. We refer the reader to the book of *Dupuis and Ellis (1997)* for a detailed discussion of the weak convergence approach to large deviations theory and some illustrative applications.
It is typical in large deviations theory the use of the notion of *exponential tightness* and the formulation of a *weak large deviations principle* in order to obtain full large deviations principles. Another technique is the transfer of large deviations principles from a given topological space to other topological spaces, through *contraction principles*. Usually these approaches rely on the use of approximations and discretizations that are difficult to pass to the limit in the respective topologies of the state spaces. For noisy perturbed dynamical

systems, the weak convergence approach totally bypasses the verification of exponential tightness and the proofs of large deviations principles reduce to the verification of basic qualitative properties of certain perturbations of the original system, such as existence, uniqueness and stability under some perturbations that lie in classical very well-studied functional spaces. As examples of the use of *the weak convergence approach* to derive large deviations results we mention the application to random walks with continuous and discontinuous coefficients (*Dupuis and Ellis (1997)*- chapters 6 and 7), the study of large deviations for Markov chains (*Dupuis and Ellis (1997)*- chapter 8), weakly interacting processes (*Budhiraja et al. (2012)*), the Brownian case (*Budhiraja and Dupuis (2000)*) and for infinite dimensional systems perturbed by a Brownian motion (*Budhiraja et al. (2008)*).
We follow *Budhiraja et al. (2011)* where the authors establish a variational formula for functionals of Poisson random measures and derive a sufficient condition for a large deviations principle for Lévy-driven dynamical systems and the subsequent work *Budhiraja et al. (2015)*, where the authors use the variational formula for functionals of Poisson random measures to derive a sufficient condition for a moderate deviations principle for dynamical systems perturbed by a Poissonian source of noise. The sufficient condition for a large deviations principle obtained in *Budhiraja et al. (2011)* was sucessfully used to the study of perturbed dynamical systems in finite dimensions (see *Budhiraja et al. (2011)* and *Budhiraja et al. (2013)*) and in infinite dimensions (*Budhiraja et al. (2013)*). We cite the work *Dong et al. (2015)* where the sufficient condition for a moderate deviations principle obtained in *Budhiraja et al. (2015)* was used to state a moderate deviations principle for the two dimensional stochastic Navier Stokes equations perturbed with multiplicative Lévy noises. Another reference is *Budhiraja and Wu (2015)* where the authors studied moderate deviations asymptotics for a certain class of particle systems.
In **Appendix D** the reader will find a detailed survey of the large/moderate deviations results that are explicitly and implicitly used throughly this work, including the ones that were mentioned briefly in the paragraphs above.

0.1 The first exit time problem for exponentially light jump diffusions

Let $U : \mathbb{R}^d \longrightarrow \mathbb{R}$ be smooth enough with a global point of minimum $0 \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. We consider a gradient dynamical system perturbed in low intensity, $\varepsilon > 0$, by a compensated compound Poisson process $(\tilde{L}^\varepsilon_t)_{t\geq 0}$, described by the stochastic differential equation,

$$\begin{cases} dX^{\varepsilon,x}_t &= -\nabla U(X^{\varepsilon,x}_t)dt + \varepsilon d\tilde{L}^\varepsilon_t, \quad t \geq 0, \\ X^{\varepsilon,x}_0 &= x. \end{cases}$$

For fixed $\varepsilon > 0$, the stochastic perturbation $(\tilde{L}_t^\varepsilon)_{t\geq 0}$ is given by

$$\tilde{L}_t^\varepsilon = \int_0^t \int_{\mathbb{R}^d} z\tilde{N}^{\frac{1}{\varepsilon}}(ds, dz),$$

where $\tilde{N}^{\frac{1}{\varepsilon}}$ is a compensated Poisson random measure defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with compensator $\varepsilon^{-1}ds \otimes \nu$. The measure ν has the form

$$\nu(dz) = e^{-|z|^\alpha}dz, \qquad \text{for some } \alpha > 0.$$

If $\alpha \geq 1$, we call ν a *superexponential jump measure* and if $\alpha \in (0,1)$ we denote ν a *subexponential jump measure*.
We prove in **Chapter 2**, under suitable assumptions, that the exit of $(X_t^{\varepsilon,x})_{t\geq 0}$ from a ball of radius $R > 0$ centered in the origin has an asymptotic upper bound, when $\varepsilon \to 0$, of the order $e^{-\varepsilon^\alpha}$, i.e. there exists some constant $C(R) > 0$ such that, for $\varepsilon > 0$ small enough, we have

$$\mathbb{P}\Big(\sup_{t\geq 0} |X_t^{\varepsilon,x}| \geq R\Big) \leq e^{-\frac{C(R)}{\varepsilon^\alpha}}.$$

In conclusion, in the superexponential regime, if $\alpha \geq 1$, the asymptotics of the exit from the ball of radius $R > 0$ and centered in the stable state of the underlying dynamical system $0 \in \mathbb{R}^d$ follow a large deviations scale. If ν is a subexponential light measure, the asymptotics of the exit from the ball follow a moderate deviations scale.
Fixed $T > 0$, $x \in \mathbb{R}^d$ and $\alpha \geq 1$, we prove in **Chapter 2** that $(X^{\varepsilon,x})_{t\in[0,T]}$ obeys a large deviations principle in the *Skorokhod space* $\mathbb{D}([0,T], \mathbb{R}^d)$ with rate function $\mathbb{J} : \mathbb{D}([0,T], \mathbb{R}^d) \longrightarrow [0, \infty]$, given by

$$\begin{aligned}
&\mathbb{J}(\varphi) := \\
&\inf\Big\{\int_0^T \int_{\mathbb{R}^d} (g(s,z)\ln g(s,z) - g(s,z) + 1)\nu(dz)ds \mid g : [0,T] \longrightarrow [0,\infty) \text{ measurable:} \\
&\varphi(t) = x - \int_0^t \nabla U(\varphi(s))ds + \int_0^t \int_{\mathbb{R}^d} z(g(s,z) - 1)\nu(dz)ds, \quad t \in [0,T]\Big\}.
\end{aligned}$$

We consider $D \subset \mathbb{R}^d$ a bounded domain satisfying some suitable conditions and such that $0, x \in D$. It is the main object of study of **Chapter 2** the asymptotics, as $\varepsilon \to 0$, of the law and the expected value of

$$\sigma^\varepsilon(x) = \inf\{t \geq 0 \mid X_t^{\varepsilon,x} \notin D\}.$$

We define the potential associated to D

$$\begin{aligned}
\bar{V} := \inf_{z\notin D} \inf_{T>0} \inf_{\varphi\in C([0,T],\mathbb{R}^d):\varphi(T)=z} \inf\Big\{&\int_0^T \int_{\mathbb{R}^d} (g(s,z)\ln g(s,z) - g(s,z) + 1)\nu(dz)ds \mid \\
&g : [0,T] \longrightarrow [0,\infty) \text{ measurable such that for } t \in [0,T] \\
&\varphi(t) = x - \int_0^t \nabla U(\varphi(s))ds + \int_0^t \int_{\mathbb{R}^d} z(g(s,z) - 1)\nu(dz)ds\Big\}.
\end{aligned}$$

If $\nu(dz) = e^{-|z|^\alpha}dz$ for some $\alpha \geq 1$, we prove in **Chapter 2** that for every $\delta > 0$, the law of the first exit time and the expected first exit time follow the asymptotics

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big(e^{\frac{\bar{V}-\delta}{\varepsilon}} \leq \sigma^\varepsilon(x) \leq e^{\frac{\bar{V}+\delta}{\varepsilon}}\Big) = 1 \text{ and}$$
$$\lim_{\varepsilon \to 0} \mathbb{E}\varepsilon[\sigma^\varepsilon(x)] = \bar{V}.$$

In **Chapter 3** we cover the study of the first exit time problem in the subexponential regime, when $\alpha \in (0,1)$. Fixed $T > 0$, and $x \in \mathbb{R}^d$, we prove that $(X_t^{\varepsilon,x})_{t\in[0,T]}$ satisfies a moderate deviations principle in the *Skorokhod space* with speed ε^α and rate function $\tilde{\mathbb{I}}_0 : \mathbb{D}([0,T], \mathbb{R}^d) \longrightarrow [0, \infty]$, given by

$$\tilde{\mathbb{I}}_0(\eta) = \begin{cases} 0, & \text{if } \eta = X^{0,x}, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$X_t^{0,x} = x - \int_0^t \nabla U(X_s^{0,x})ds, \quad t \geq 0,$$

is the unperturbed dynamical system. If we construct the potential associated with the rate function $\tilde{\mathbb{I}}_0$ and the bounded domain D, with $0, x \in D$, under suitable assumptions that make the trajectories of the dynamical system described by $(X_t^{0,x})_{t\geq 0}$ being contained in D, we conclude that

$$\bar{V}_0 = \infty.$$

For this reason, we study the first exit time problem for the renormalized deviation of the stochastic perturbed dynamical system $(X_t^{\varepsilon,x})_{t\geq 0}$ from the deterministic one $(X_t^{0,x})_{t\geq 0}$,

$$Y_t^{\varepsilon,x} := \frac{X_t^{\varepsilon,x} - X_t^{0,x}}{a(\varepsilon)},$$

where $a(\varepsilon) = \varepsilon^{\frac{1-\alpha}{2}}$. Fixed $x \in \mathbb{R}^d$, we prove that the family $(Y^{\varepsilon,x})_{\varepsilon>0}$ satisfies a moderate deviations principle with speed ε^α and rate function $\tilde{\mathbb{I}}_1 : \mathbb{D}([0,T], \mathbb{R}^d) \longrightarrow [0, \infty]$ given by

$$\begin{aligned} \tilde{\mathbb{I}}_1(\eta) := \inf\Big\{ &\frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |\psi(s,z)|^2 \nu(dz)ds \mid \psi \in L^2(ds \otimes \nu) \text{ such that} \\ &\eta(t) = -\int_0^t \nabla^2 U(X_s^{0,x})ds + \int_0^t \int_{\mathbb{R}^d} z\psi(s,z)\nu(dz)ds, \quad t \in [0,T]\Big\}. \end{aligned}$$

The main object of study of **Chapter 3** is the asymptotic study of the first exit time of $(Y_t^{\varepsilon,x})_{t\geq 0}$ from D smooth enough such that $0, x \in D$, as $\varepsilon \to 0$,

$$\tilde{\sigma}^\varepsilon(x) := \inf\{t \geq 0 \mid Y_t^{\varepsilon,x} \notin D\}, \text{ for } x \in D.$$

We define the potential

$$\bar{V}_1 = \inf_{z\notin D} \inf_{T>0} \inf_{\eta\in C([0,T],\mathbb{R}^d):\eta(T)=z} \inf \Big\{ \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |\psi(s,z)|^2 \nu(dz)ds \mid \psi \in L^2(ds\otimes\nu) \text{ such that}$$
$$\eta(t) = -\int_0^t \nabla^2 U(X_s^{0,x})ds + \int_0^t \int_{\mathbb{R}^d} z\psi(s,z)\nu(dz)ds, \quad t\in[0,T] \Big\}.$$

In **Chapter 3** we prove the following asymptotic results for $\tilde{\sigma}^\varepsilon(x)$, for $x \in D$ and $\delta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(e^{\frac{\bar{V}_1-\delta}{\varepsilon^\alpha}} \leq \tilde{\sigma}^\varepsilon(x) \leq e^{\frac{\bar{V}_1+\delta}{\varepsilon^\alpha}} \Big) = 1 \text{ and}$$
$$\lim_{\varepsilon\to 0} \varepsilon^\alpha \mathbb{E}[\tilde{\sigma}^\varepsilon(x)] = \bar{V}_1.$$

In conclusion, in the subexponential regime, the first time that $Y^{\varepsilon,x}$ exits the domain follows a speed rate in the moderate deviations regime of order ε^α according to a rate function that has a quadratic form. In the superexponential regime the first exit time has an asymptotic rate of order ε following a large deviations scale with a Poissonian rate function. In both scenarios the perturbed system will experiment every possible path with probabilities that are exponentially small. The higher the rate function, which means that the path is less efficient, the smaller is the probability of occurrence and therefore, the less frequent the attempt to escape the domain following that path. In **Chapter 1** the reader will find the results that are mentioned stated rigorously and in detail. In **Chapter 2** and **Chapter 3** we address the first exit time problem in the superexponential and subexponential regime respectively.

Related literature

We refer the reader to the classic book *Freidlin-Wentzell (1988)* where the authors apply large deviations results to the study of the first exit time of dynamical systems perturbed by Brownian noise and to further metastability results for the perturbed dynamics. If $(B_t)_{t\geq 0}$ is a Brownian motion, $U : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with a minimum point $0 \in [a, b]$, $0 < a < b < \infty$ and $U(a) < U(-b)$ we consider the one dimensional perturbed dynamical system in low intensity $\varepsilon \to 0$,

$$X_t^{\varepsilon,x} = x - \int_0^t U'(X_s^{\varepsilon,x})ds + \varepsilon B_t, \quad t \geq 0.$$

The *Freidlin-Wentzell theory* shows the exponential decay of order ε^{-2} of the law of the probability of exit in terms of the potential barrier $U(a)$ where the exit is privileged. Consequently, under suitable assumptions, The *Freidlin Wentzell theory* expresses the expected first time of exit in terms of the evaluation of the potential in the privileged boundary point of the domain with some geometrical pre-factor that depends on the curvature of the potential in the boundary point where the exit is privileged and on the slope at the local minimum of the potential, that is respectively the place where the perturbed Brownian

diffusion stays more time before exiting the domain. This is known in Physics literature as *Kramer's law. Kramer's law* has its origin in the study of chemical reactions. We refer to *Kramer (1940)* and to *Kampen (1981)* for details.
First exit times results were derived for other Lévy-driven dynamical systems where the stochastic source of perturbation is an alpha-stable process. In *Imkeller and Pavlyukevich (2006)* the authors studied the first exit time of a perturbed one-dimensional gradient dynamical system by an alpha-stable process, a pure jump process with Lévy measure $\nu(dz) = \frac{dz}{|z|^{1+\alpha}}$, for some $\alpha \in (0, 2)$ in the small noise limit. The authors concluded that the asymptotics of the expected first exit time from a given compact interval follows, as $\varepsilon \to 0$, a polynomial scale, $\frac{1}{\varepsilon^{\alpha}}$ with some prefactor that depends on the distance beween the boundary points of the interval from the stable state of the underlying dynamical system. Due to the presence of the large jumps the time the jump diffusion has to escape from the domain is much shorter (polynomial scale $\varepsilon^{-\alpha}$) in comparison with the Brownian case, where the time to escape from the interval is exponentially large in ε^{-2}. In *Imkeller and Pavlyukevich (2008)* the authors study the metastable behaviour of these jump-diffusions, concluding that the transition times between the wells of the corresponding stable states of the underlying dynamical system follow also polynomial scales ε^{alpha} in the small noise limit. In *Pavlyukevich (2011)* the author studies the first exit time problem for a dynamical system perturbed in low intensity with multiplicative alpha-stable noise in the multidimensional setting. In *Högele and Pavlyukevich (2014)* and *Högele and Pavlyukevich (2015)* the first exit time problem and metastable behaviour studies were extended for non-gradient perturbed dynamical systems by alpha-stable processes. In *Debussche et al. (2011)* and *Debussche et al. (2013)* the authors addressed the first exit time problem and corresponding metastability results in an infinite-dimensional setting studying with detail the dynamics of the *stochastic Chaffee-Infante equation* perturbed in low intensity $\varepsilon \to 0$ by an alpha-stable process. The *stochastic Chaffee-Infante equation* is a perturbed energy balanced model for the global averaged temperature of the earth and it is an example of a perturbed dynamical system by jump noises used in climate modeling. The stochastic jump perturbation of the dynamical system capture the abbrupt changes of temperature in very small time scales in comparison with the time horizon of the study. Here the comprehension of the first exit time problem becomes of great importance in order to understand and extract some statistical information about the occurrence of those abrupt changes of temperatures. In this setting the (big) jumps of the stochastic perturbation are used to describe the rapid catastrophic climate changes which occurred in the Earth's northern hemisphere (the so called *Daansgard-Oeschger events*). We refer the work *Dietlevesen (1999)* where the author shows evidence of the discovery of an alpha-stable noise signal, with the heaviness parameter $\alpha \simeq 1,75$; *Hein et al. (2009)* where the authors study calibrations and the p-variations of α-signals encountered in paleoclimatic data and *Gairing et al. (2016)* where the authors confront theoretical results on transport distances for Lévy processes and certain paleoclimatic time series. We refer to *Imkeller and Monahan (2002)* and *Dijkstra (2013)* for an account of the use of stochastic analysis on climate dynamics.

In *Imkeller et al. (2009)* the authors studied the first exit time problem for a one-dimensional gradient system perturbed by exponentially light jump processes. Specifically it was considered, for a certain smooth function $U : \mathbb{R} \longrightarrow \mathbb{R}$ under suitable assumptions, the following SDE

$$\begin{cases} dX_t^\varepsilon & = -U'(X_t^\varepsilon) + \varepsilon dL_t \\ X_0^\varepsilon & = x, \end{cases}$$

where the Lévy measure has exponentially light tails, $\nu[u, \infty) \simeq e^{-u^\alpha}$, when $u \to \infty$, for some $\alpha > 0$. The authors concluded the asymptotic behaviour of the expected first exit time with a continuous phase transition in $\alpha = 1$ of the form

$$\varepsilon^{-\alpha} \quad \text{if } \alpha \in (0,1) \quad \text{and} \quad \varepsilon^{-1} |\ln \varepsilon|^{1-\frac{1}{\alpha}}, \quad \text{if } \alpha \geq 1.$$

We remark that the results mentioned above for the alpha-stable case and for the exponentially light jump perturbation rely on a technique of decomposing the noise component into small jumps and big jumps for some threshold. The results that were surveyed above correspond to stochastic perturbations of the type $(\varepsilon L_t)_{t \geq 0}$. In our work we deal with stochastic perturbations of the type $(\varepsilon \tilde{L}_t^\varepsilon)_{t \geq 0}$ since we rescale the intensity measure of the underlying Poisson random measure by a factor $\frac{1}{\varepsilon}$. We mention also the Diploma thesis of *Hinze (2010)* that describes which properties a symmetric Lévy measure must fulfill such that Gaussian exit is attainable for a gradient dynamical system perturbed by a jump process. *Hinze* concludes that a perturbation of the form $(\varepsilon L_t)_{t \geq 0}$ is not enough to oberve exit of the form $e^{-\varepsilon^2}$ as $\varepsilon \to 0$ from a fixed bounded interval. The author gives also lower and upper bounds for the mean time of exit. Nevertheless, he does not write the asymptotics of the law of the first exit time or the mean first time of exit in function of a given potential or fixed quantity in contrast with this work. Another difference is that we study the first exit time problem for the multidimensional case.

0.2 The small noise limit for a forward-backward system of SDEs with jumps

Lévy flights is a popular term in Physics for random walks in which the step lenghts U have a heavy-tailed distribution, i.e. $\mathbb{P}(U > u) = O(u^{-\alpha})$ for some $\alpha \in (1,2)$. They are appropriate models that capture non Gaussian effects and where diffusive behavior is not adequate. Their use is well-known in climate modeling, animal hunting patterns and in the modeling of molecular gases in non-homogeneous media. We refer the reader to *Sokolov (2012)* and *Klafter and Metzler (2004)* for further references.
Let us fix a terminal time $T > 0$. If we consider a system of particles whose motion is governed by *Lévy flights* and perform the hydrodynamic limit, in the presence of some additional assumptions, we end up with the so-called fractal Burgers Equations, where ν is the viscosity parameter,

$$\begin{cases} \partial_t v^\nu(t,x) = -\nu(-\Delta)^{\frac{\alpha}{2}} v^\nu(t,x) - \langle v^\nu(t,x), \nabla_x v^\nu(t,x) \rangle + F^\nu(t,x) = 0, \\ v^\nu(0,x) = g(x), \quad t \in [0,T], \quad x \in \mathbb{R}^d. \end{cases}$$

The solution v^ν of the fractal Burgers equations models the velocity of a compressible fluid with nonlocal viscosity parameter $\nu > 0$ that shows a fractional (nonlocal) diffusive behavior captured by the presence of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (0, 2)$, and affected by a force F^ν that captures local and non-local sources of interaction depending eventually on the velocity of the fluid itself. We stress that this semilinear term F^ν is not stochastic. The initial condition g is the initial configuration of the velocity field in all space $\mathbb{R}^d$. The fractional Laplacian is an integral-differential operator defined by

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = c_{d,\alpha} \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \frac{|f(x) - f(y)|}{|x-y|^{d+\alpha}} dy,$$

for all the measurable functions f whenever the limit above exists and is well-defined. The constant $c_{d,\alpha}$ is defined by

$$c_{d,\alpha} := \frac{\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2^{1-\alpha} \pi^{d-2} \Gamma\left(1 - \frac{\alpha}{2}\right)},$$

where Γ is *Euler's Gamma function.*

The presence of $(-\Delta)^{\frac{\alpha}{2}}$ in the structure of the equations is not surprising since, via the *Kolmogorov functional limit theorem*, the distance from the origin of the *Lévy flights* converges, after a large number of steps, to an α-stable law and $(-\Delta)^{\frac{\alpha}{2}}$ is the infinitesimal generator of an α-stable process.

We do not enter in details for the functional study of this operator and refer the reader to *Di Nezza et al. (2012).* The fractal Burgers equations form an example of a system of partial-integral differential equations (PIDEs for short). PIDEs are a preeminent topic of active research in mathematics with the growing demand of the use of differential equations that take into account nonlocal effects of interaction and non-isotropic propagation of energy. Fractal Burgers equations increased interest in models involving fractional dissipation, in particular in Navier-Stokes equations (see *Katz and Pavlović (2002)*), combustion models (see *Matalon (2007)*) and the surface geostrophic equation (see *Constantin et al. (2001)*). These equations have been studied in *Biler et al. (1998)* and in *Aschterenberg et al. (2008). Zhang (2012)* studies probabilistically the fractal Navier Stokes equation which turns as an example in favor of probabilistic approaches to the study of nonlocal hydrodynamic models, as was made before to the Navier Stokes systems. We refer the reader to *Cruzeiro and Shamarova (2009)*, *Constantin and Iyer (2008)* and *Busnelo et al. (2005)* as examples of probabilistic studies of Navier-Stokes equations. We will associate a certain class of partial-integral differential equations, including the fractal Burgers equation, with a certain system of stochastic differential equations and via this probabilistic object we will address the problem of the vanishing viscosity limit $\nu \to 0$.

Fix $T > 0$ and $\varepsilon, \delta > 0$. Consider the following functions

$$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \text{and} \quad \beta : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

We assume that the functions f and β are smooth with bounded derivatives of all orders. Onder these assumptions we can state that, for every $\varepsilon, \delta > 0$, the following terminal value problem for the fractal Burgers equation (with also local diffusive component given by $\frac{\varepsilon}{2}\Delta$) has a smooth solution,

$$\begin{cases} \partial_t u^{\varepsilon,\delta}(t,x) + \langle \nabla_x u^{\varepsilon,\delta}(t,x), u^{\varepsilon,\delta}(t,x)\rangle + \frac{\varepsilon}{2}\Delta u^{\varepsilon,\delta}(t,x) + \delta(-\Delta)^{\frac{\alpha}{2}} u^{\varepsilon,\delta}(t,x) \\ +f\Big(t, x, u^{\varepsilon,\delta}(t,x), \varepsilon\nabla_x u^{\varepsilon,x}(t,x), u^{\varepsilon,\delta}(t, x+\delta\beta(x,.)) - u^{\varepsilon,\delta}(t,x)\Big) = 0, \\ u^{\varepsilon,\delta}(T,x) = g(x), \quad t \in [0,T], \quad x \in \mathbb{R}^d. \end{cases}$$

Here, the parameter ε is the local viscosity parameter and δ is the nonlocal viscosity parameter. The semilinear term f captures local and nonlocal sources of interactions in the evolution of the velocity field $u^{\varepsilon,\delta}$ and β is a displacement function in space.
Let us denote the solution of the terminal value problem above by $u^{\varepsilon,\delta}$. The function $u^{\varepsilon,\delta} \in C^{1,2}([0,T] \times \mathbb{R}^d)$ with bounded derivatives (see *Situ (1997)*).

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in which we define a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ and, for every $\delta > 0$ an independent compensated Poisson random measure $\tilde{N}^{\frac{1}{\delta}}$ with compensator given by $\frac{1}{\delta}ds \otimes \nu$ and $\nu(dz) = \frac{1}{|z|^{d+\alpha}}dz$. Fixed $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\varepsilon, \delta > 0$, since $u^{\varepsilon,\delta}$ is smooth enough *Situ (1997)* and β is C^∞ with bounded derivatives, let $(X_s^{\varepsilon,\delta})_{s\in[t,T]}$ be the unique solution of the following SDE, for all $s \in [t,T]$,

$$X_s^{\varepsilon,\delta} = x + \int_t^s u^{\varepsilon,\delta}(s, X_r^{\varepsilon,\delta})dr + \sqrt{\varepsilon}(B_s - B_t) + \delta \int_t^s \int_{\mathbb{R}^d} \beta(X_{r-}^{\varepsilon,\delta}, z)\tilde{N}^{\frac{1}{\delta}}(dr, dz).$$

Using *Itô's formula* and defining, for every $s \in [t,T]$,

$$\begin{cases} Y_s^{\varepsilon,\delta} &:= u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta}), \\ Z_s^{\varepsilon,\delta} &:= \varepsilon\nabla_x u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta}), \\ V_s^{\varepsilon,\delta} &:= u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta} + \delta\beta(X_s^{\varepsilon,\delta}, z)) - u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta}), \end{cases}$$

we can conclude that $(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, Z_s^{\varepsilon,\delta}, V_s^{\varepsilon,\delta})_{s\in[t,T]}$ solves the following system of stochastic differential equations, for every $s \in [t,T]$,

$$\begin{cases} X_s^{\varepsilon,\delta} &= x + \int_t^s u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta})ds + \sqrt{\varepsilon}(B_s - B_t) + \delta \int_t^s \int_{\mathbb{R}^d} \beta(X_{s-}^{\varepsilon,\delta}, z)\tilde{N}^{\frac{1}{\delta}}(ds, dz), \\ Y_s^{\varepsilon,\delta} &= g(X_T^{\varepsilon,\delta}) + \int_s^T f(s, X_s^{\varepsilon,\delta}, Y_r^{\varepsilon,\delta}, Z_r^{\varepsilon,\delta}, V_r^{\varepsilon,\delta})dr \\ & \quad - \int_s^T Z_r^{\varepsilon,\delta} dB_r - \int_s^T \int_{\mathbb{R}^d} V_{r-}^{\varepsilon,\delta}(z)\tilde{N}^{\frac{1}{\delta}}(dr, dz). \end{cases}$$

The representation $u^{\varepsilon,\delta}(t,x) = Y_t^{\varepsilon,\delta}$ is known in the FBSDEs literature as a nonlinear *Feynman-Kac formula.*
We consider a more general system of FBSDEs with jumps than the one presented in the example above and we study the asymptotics of the FBSDE system with a Brownian and a Poissonian component and the associated PIDE if both sources of noise are affected by parameters that vanish. Secondly, under more restrictive assumptions on the FBSDE system and specially on the Lévy measure we obtain, via a sufficient condition derived in *Budhiraja et al. (2011)*, a large deviations principle for the laws of the forward process. Finally we transfer with a contraction principle the large deviations principle of the laws of the forward process to the laws of the backward process.
The motivation to study LDPs for such systems of FBSDEs lies in the connection of the representation formulas from Calculus of Variations for the solutions of the associated nonlinear PDEs with the variational principle of *Laplace-Varadhan.* Writing solutions of parabolic PDEs as functionals of Brownian diffusions has a huge history and it is a well known fact exploited by the *Feynman-Kac* formulas. Taking the *Burgers* equations as a paradigmatic example, *Varadhan* exploited in *Varadhan (1966)* the vanishing viscosity limit for such equations via the connection of the *Cole-Hopf* transform and the variational principle for functionals of Brownian Motion that is nowadays called *Laplace-Varadhan principle.* Vanishing viscosity limits of nonlinear PDEs can be approached with large deviations principles when writing the respective solutions as functionals of the associated backward processes and identifying the limit via the variational methods that are equivalent to the *Laplace-Varadhan principles* which the backward processes obey. Therefore, this straight link between the limiting behavior of PDEs and LDT was our motivation to the comprehension of the problem we exposed before.

Related literature

Forward backward stochastic differential equations became very popular in the last twenty years due to the huge range of applications and interactions with other mathematical fields. Besides the connections with PDEs, is very well known the strong link FBSDE systems have with stochastic optimal control. Stochastic optimal control solution theory has two important methodologies: the *dynamic programming principle* and the *Pontryagin maximum principle.* The first one deals with the asociated Hamilton-Jacobi-Bellman (HJB) equation. HJB is a deterministic PDE whose solution is the value function for the stochastic optimization problem. The Pontryagin maximum principle involves the maximization of a Hamiltonian and solving the adjoint equation, which is a BSDE (see *Ma and Yong (1995)* for details). With a stochastic optimal control problem in mind, *Bismut (1973)* introduces a linear BSDE asociated to the *Pontryagin maximum principle.* General nonlinear BSDE theory in the Brownian case was developed in *Pardoux and Peng (1990).*
There are four main methods to solve FBSDEs:

i) *The contraction mapping*, which assures the existence and uniqueness of solutions in a small time interval via a Picard iteration scheme.

ii) *The four step scheme*, developed in *Ma et al. (1995)*, which although it requires more strict assumptions, such as deterministic coefficients and non-degeneracy of the matrix diffusion for the forward equation, produces an existence and uniqueness result of solution in an arbitrarily large time interval. Here the backward process is a function of the forward process via the associated HJB equation.

iii) *The method of continuation*, investigated in *Peng and Wu (1999)* and in *Yong (1997)*, that allows the FBSDEs systems to have random coefficients.

iv) *The method of decoupling fields*, that searches not only for a solution of the FBSDE system, but also for some measurable function, denominated decoupling field, that expresses the backward process in terms of the forward one, solving the FBSDE and finding respective decoupling fields in small intervals. In a second step the method searches for a a global solution for the FBSDE on the time interval via the concatenation of the decoupling fields. We refer the reader to the thesis of *Fromm (2014)* where the decoupling fields method is presented in detail for the Brownian case.

We refer the reader to the books *Ma and Young (1999)* and *Pardoux and Răscanu (2014)* for the solution theories for FBSDEs in the Brownian case and to *Delong (2013)* for the jump case. As a natural generalization, FBSDEs driven by jump diffusions became an increasingly popular and natural object of study. BSDEs with jumps were discussed in *Li and Tang (1999)* and their connections with viscosity solutions of the associated system of parabolic integral-differential equations were first discussed in *Barles et al. (1996)*.
Concerning large deviations statements for FBSDES we mention for the Brownian case *Rainero (2006)*, where the same problem was addressed for a decoupled FBSDE system (i.e. the forward equation does not depend on the backward process); *Cruzeiro et al. (2014)* for the Brownian coupled case; and *Frei and Reis (2013)* where the authors studied the vanishing viscosity limit of diffusive quadratic Burgers nonlinearities via FBSDEs. For the jump case we mention *Sow (2014)* where the FBSDE that is addressed is not coupled with non-Lipschitz coefficients. Our studies cover a certain kind of coupled FBSDE systems that includes in the class of the associated quasilinear PIDEs the example of the fractal Burgers equations, discussed in the last paragraph.

0.3 Organization of the work

In **Chapter 1** we state the main results concerning the first exit time problem for exponentially light jump diffusions. In **Chapter 2** and **Chapter 3** we address the first exit time problem for exponentially light jump processes covering the superexponential case in **Chapter 2** and respectively the subexponential regime in **Chapter 3**.
In **Chapter 4** we address the small noise limit of a forward backward system of stochastic differential equations with jumps that covers the example that we stated in the introduction. We present a result of existence and uniqueness of solution in a small time interval

and we explore the connections of these probabilistic objects with viscosity solutions of certain PIDEs. We study the almost sure convergence of the FBSDE system and the convergence at the level of the viscosity solutions of the respective PIDE if the intensity parameters of the Browian and Poissonian components of the noise vanish. Finally we derive a large deviations principle for the laws of the solution processes of the considered FBSDE in the small noise limit.
In the **Appendix A** we collect some standard results and facts that are used along the text. **Appendix B** and **Appendix C** contain a small collection of definitions and results concerning Lévy processes, Poisson random measures and some generic classical facts about weak convergence of probability measures and the *Skorokhod space*. It is the intention of **Appendix D** to be a concise survey of large deviations theory results that we use along our work. We state the generic tools and results from large deviations in order to prove the main theorems of **Chapter 1**. We follow closely *Budhiraja et al. (2011)* and *Budhiraja et al. (2015)* in the presentation of the variational principle for functionals of Poisson random measures and in the proofs of the sufficient conditions for large/moderate deviations principles for dynamical systems driven by Poisson random measures that we use in this work.

Chapter 1

The main results for the first exit time problem

The main task of this work consists in the establishment and the applications of large deviations principles to study two different problems. First, in **Chapter 2** and **Chapter 3** we study the first exit time problem for a certain class of perturbed dynamical systems at low intensity by a specific type of Lévy processes and secondly in **Chapter 4** the small noise limit of a forward backward system of stochastic differential equations with jumps. The equalities and inequalities between random variables are to be understood in the almost sure sense.

In this chapter we state the main results concerning the first exit time problem for exponentially light diffusions that was described in the introduction. We start with the definition of large deviations principle.

In a topological space $\mathcal{S}$, given a set $A \subset \mathcal{S}$, $\mathrm{int}A$ and $\mathrm{cl}A$ stand for the topological interior and topological closure of the set A.

Definition 1.0.1 (Large deviations principle). *Let us fix a topological space $\mathcal{S}$ and a function $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying $b(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

i) Let $I : \mathcal{S} \longrightarrow [0, \infty]$ be a function such that, for every $a \geq 0$, the sublevel set

$$\{x \in \mathcal{S} \mid I(x) \leq a\} \quad \textit{is compact.}$$

*We call I a **good rate function**.*

*ii) A family $(X^\varepsilon)_{\varepsilon>0}$ of $\mathcal{S}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy a **large deviations principle with speed** $b(\varepsilon)$ in $\mathcal{S}$ and with good* *rate function I if for every Borel set $A \in \mathcal{B}(\mathcal{S})$*

$$\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in A) \leq -\inf_{x\in clA} I(x),$$
$$\liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in A) \geq -\inf_{x\in intA} I(x).$$

iii) *If $b(\varepsilon) = \varepsilon$, we say simply that $(X^\varepsilon)_{\varepsilon>0}$ satisfies a* ***large deviations principle*** *with good rate function I.*

iv) *If $b(\varepsilon) := \frac{\varepsilon}{a^2(\varepsilon)}$, for some function $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $a(\varepsilon) \to 0$ as $\varepsilon \to 0$, we say that the family $(X^\varepsilon)_{\varepsilon>0}$ satisfies a* ***moderate deviations principle*** *with good rate function I.*

We use the convention that $\inf \emptyset = \infty$.

Remark 1.0.1.

i) *It is usual in the literature that the definition of large deviations principle only asks the functional $I : \mathcal{S} \to [0,\infty]$ to be lower semicontinuous, i.e. a rate function. We ask instead that I is a good rate function.*

ii) *If $\mathcal{S}$ is a regular Hausdorff space the good rate function I associated to the large deviations principle of $(X^\varepsilon)_{\varepsilon>0}$ with speed b of* ***Definition 1.0.1*** *is unique. This is proved in* ***Proposition D.1.1****.*

iii) *In our work $\mathcal{S}$ is usually a Polish space, i.e. a separable completely metrizable topological space, such as the space of càdlàg functions (right continuous with left limits) $\mathbb{D}([0,T],\mathbb{R}^d)$ equipped with the Skorokhod topology. The definition of Skorokhod topology is given in the next section and in the* ***Appendix C*** *the reader will find more details about this space.*

iv) *For a self-contained presentation of the large deviations results available in the literature and that are used in our work we refer the reader to* ***Appendix D****.*

1.1 The mathematical framework

Fix $x \in \mathbb{R}^d$. We consider the dynamical system described by the following ordinary differential equation,

$$u(t;x) = x - \int_0^t \nabla U(u(s;x))ds, \quad t \geq 0. \tag{1.1.1}$$

Condition 1.1.1 (**Assumptions on the unperturbed dynamical system**).

i) Let $U : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^2 function. The point $0 \in \mathbb{R}^d$ satisfies $U(0) = 0$, $\nabla U(0) = 0$ and the Hessian matrix $-\nabla^2 U(0)$ is negative definite. Furthermore, there exists $\eta > 0$ such that

$$\langle -\nabla U(x) + \nabla U(y), x - y \rangle \leq -\eta |x - y|^2 \quad \text{for all } x, y \in \mathbb{R}^d. \tag{1.1.2}$$

ii) We fix a bounded domain $D \subset \mathbb{R}^d$ such that $0 \in D$. We assume the following conditions on $u(.;x)$:

$$\text{For every } x \in \mathbb{R}^d \lim_{t\to\infty} u(t;x) = 0.$$
$$\text{If } x \in cl(D) \text{ then } u(t;x) \in D, \quad t \geq 0.$$

Under **Condition 1.1.1**, for fixed $x \in \mathbb{R}^d$, it is a well-known fact that there exists a unique continuous function $u(.;x) : [0,\infty) \longrightarrow \mathbb{R}^d$ that satisfies the equation above for all $t \geq 0$.

We perturb (1.1.1) with small intensity $\varepsilon > 0$ by a specific Lévy process. We describe the probability space where the stochastic perturbation is defined.

Let us fix a non-atomic locally finite measure ν defined on the Borel sets of $\mathbb{R}^d$, i.e. a measure such that $\nu(\{z\}) = 0$ for all $z \in \mathbb{R}^d$ and $\nu(K) < \infty$ for all compact sets $K \subset \mathbb{R}^d$. We denote by $\mathbb{M}$ the space of the locally finite measures defined on the Borel sets of $[0,\infty) \times \mathbb{R}^d$.
We consider the Cartesian product $[0,\infty) \times \mathbb{R}^d \times [0,\infty)$. Given a Poisson randon measure defined on the Borel sets of $[0,\infty) \times \mathbb{R}^d$, the enlargement of $[0,\infty) \times \mathbb{R}^d$ with a third component space has the following role: the first component takes into account the time variable t; the second one is the space of the jumps z of the process associated to a Poisson random measure; and the third one registers the frequencies r of the jumps.
Let us denote by $\bar{\mathbb{M}}$ the space of the locally finite measures defined on the Borel measurable space $([0,\infty) \times \mathbb{R}^d \times [0,\infty), \mathcal{B}([0,\infty) \times \mathbb{R}^d \times [0,\infty)))$.

Due to **Proposition B.2.1** there exists a unique probability measure $\bar{\mathbb{P}}$ defined on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$ such that the canonical map

$$\bar{N} : \bar{\mathbb{M}} \longrightarrow \bar{\mathbb{M}},$$

$$\bar{N}(\bar{m}) := \bar{m}$$

is a Poisson random measure defined on the probability space $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ with intensity measure $ds \otimes \nu \otimes dr$, where dr denotes the Lebesgue measure on the third component of the Cartesian product $[0,\infty) \times \mathbb{R}^d \times [0,\infty)$. We denote the expectation operator with respect to $\bar{\mathbb{P}}$ by $\bar{\mathbb{E}}$.
Given $\varepsilon > 0$, we note that a Poisson random measure $N^{\frac{1}{\varepsilon}}$ of intensity $\frac{1}{\varepsilon} ds \otimes \nu$ can be represented as a controlled random measure in the following way: for $t \geq 0$ and $U \in \mathcal{B}(\mathbb{R}^d)$,

$$N^{\frac{1}{\varepsilon}}([0,t] \times U) = \int_0^t \int_U \int_0^\infty \mathbf{1}_{[0,\frac{1}{\varepsilon}]}(r) \bar{N}(ds, dz, dr). \tag{1.1.3}$$

For more details we refer the reader to **Subsection D.5.1** and **Subsection D.5.3.** of the **Appendix**.

For every $\varepsilon > 0$ and given $x \in \mathbb{R}^d$ we consider the following stochastic differential equation,

$$X_t^{\varepsilon,x} = x - \int_0^t \nabla U(X_s^{\varepsilon,x}) ds + \varepsilon \int_0^t \int_{\mathbb{R}^d} z \tilde{N}^{\frac{1}{\varepsilon}}(ds, dz), \quad t \geq 0. \tag{1.1.4}$$

For given $x \in \mathbb{R}^d$, we treat (1.1.4) as a stochastic perturbation of the dynamical system described by (1.1.1). For this reason often we write often $(X_t^{0,x})_{t \geq 0}$ instead of $u(.;x)$ for the solution of (1.1.1).

Condition 1.1.2. *For every $\varepsilon > 0$, the stochastic process*

$$\tilde{L}_t^\varepsilon := \int_0^t \int_{\mathbb{R}^d} z \tilde{N}^{\frac{1}{\varepsilon}}(ds, dz), \quad \textit{for all } t \geq 0,$$

is a compensated compound Poisson process written as a stochastic integral with respect to the compensated Poisson random measure $\tilde{N}^{\frac{1}{\varepsilon}}$. The compensator of $\tilde{N}^{\frac{1}{\varepsilon}}$ is $\frac{1}{\varepsilon} ds \otimes \nu$.

For every $t \geq 0$ we define the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by the Poisson random measure $N = N^1$ with intensity $ds \otimes \nu$, given via the representation by (1.1.3); i.e. for every $t \geq 0$,

$$\mathcal{F}_t := \sigma\Big\{ \bar{N}([0,s] \times A \times C) \mid s \leq t, \quad A \in \mathcal{B}(\mathbb{R}^d), \quad C \in \mathcal{B}([0,\infty)) \Big\}.$$

Let $(\bar{\mathcal{F}}_t)_{t \in [0,T]}$ be the completion of $(\mathcal{F}_t)_{t \geq 0}$ with respect to the probability measure $\bar{\mathbb{P}}$.
For every $T > 0$, we denote by $\mathbb{D}([0,T], \mathbb{R}^d)$ the space of the functions $f : [0,T] \longrightarrow \mathbb{R}^d$ that are right continuous and have left-limits. For more details about this space and the topology with which it is endowed we refer the reader to **Section C.2** of the Appendix.

Definition 1.1.1 (Solution of (1.1.4)). *Let $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ be the probability space introduced in the beginning of this section, where $\tilde{N}^1$ is the compensated Poisson random measure with compensator $ds \otimes \nu$.*
For every $T > 0$ and $\varepsilon > 0$, a stochastic process $(X_t^{\varepsilon,x})_{t \in [0,T]}$ defined on the probability space $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ is said to be a strong solution of (1.1.4) with initial value $x \in \mathbb{R}^d$ if we have

i) $(X^{\varepsilon,x}_t)_{0\le t\le T}$ *is* $(\bar{\mathcal{F}}_t)_{0\le t\le T}$ *adapted;*

ii) $(X^{\varepsilon,x}_t)_{0\le t\le T} \in \mathbb{D}([0,T],\mathbb{R}^d)$ $\bar{\mathbb{P}}$*- a.s. ;*

iii) $(X^{\varepsilon,x}_t)_{0\le t\le T}$ *solves (1.1.4) for all* $t \in [0,T]$ $\bar{\mathbb{P}}$*- a.s.*

The following result is standard from the theory of stochastic differential equations driven by Lévy processes. We refer to *Ikeda and Watanabe (1981)*-**Theorem IV.9.1** for a proof. For sake of completeness of the text, since our assumptions on the coefficients differ from the ones presented in the mentioned reference, we sketch a proof in **Section B.4** of the Appendix.

Theorem 1.1.1 (**Existence and uniqueness of solution of (1.1.4)**). *Given* $\varepsilon > 0$*,* $T > 0$ *and* $x \in \mathbb{R}^d$*, under* ***Condition 1.1.1*** *and* ***Condition 1.1.2*** *there exists a measurable map* $\mathcal{G}^{\varepsilon,x} : \mathbb{M} \longrightarrow \mathbb{D}([0,T],\mathbb{R}^d)$ *such that* $X^{\varepsilon,x} = \mathcal{G}^{\varepsilon,x}(\varepsilon\tilde{N}^{\frac{1}{\varepsilon}})$ $\bar{\mathbb{P}}$*-a.s. is an* $(\bar{\mathcal{F}}_t)_{0\le t\le T}$*-adapted process solving uniquely the SDE (1.1.4) in the sense of* ***Definition 1.1.1****. Furthermore,* $T = \infty$*.*

Given $\varepsilon > 0$ and τ a $(\bar{\mathcal{F}}_t)_{t\ge 0}$-stopping time, we define the σ-algebra of the past of τ,

$$\bar{\mathcal{F}}_\tau := \{A \in \mathcal{B}(\bar{\mathbb{M}}) \mid \{\tau \le t\} \cap A \in \bar{\mathcal{F}}_t \text{ for all } t \ge 0\}.$$

For every $\varepsilon > 0$ and $x \in \mathbb{R}^d$, let $(X^{\varepsilon,x}_t)_{t\ge 0}$ be the solution of (1.1.4) in the sense of **Definition 1.1.1**. We define the Markov semigroup on the space $\mathbb{M}_b(\mathbb{R}^d)$ of the bounded measurable functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$,

$$(P_t f)(x) := \bar{\mathbb{E}}[f(X^{\varepsilon,x}_t)], \quad \text{for } t \ge 0, x \in \mathbb{R}^d.$$

The family $(P_t)_{t\ge 0}$ is a contracting C_0-semigroup on $\mathbb{M}_b(\mathbb{R}^d)$. For details, we refer for instance to *Applebaum (2009)* - **Theorem 3.12** and *Applebaum (2009)*-**Chapter 7**.

Proposition 1.1.1 (**Strong Markov property**). *Under the conditions of* ***Theorem 1.1.1****, we consider the solution* $(X^{\varepsilon,x})_{t\ge 0}$ *of (1.1.4) for* $\varepsilon > 0$ *and* $x \in \mathbb{R}^d$*. Then for any* $(\bar{\mathcal{F}}_t)_{t\ge 0}$*- stopping time* τ *such that* $\bar{\mathbb{P}}(\tau < \infty) = 1$*, we have the following identity:*

$$\bar{\mathbb{E}}\Big[f(X^{\varepsilon,x}_{\tau+t})|\bar{\mathcal{F}}_\tau\Big] = (P_t f)(X^{\varepsilon,x}_\tau), \quad \textit{for all } t \ge 0, f \in \mathbb{M}_b(\mathbb{R}^d).$$

We refer the reader to *Protter (2005)*-**Theorem 32**.

The jump measure ν. Given $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we study the problem of the first exit of $(X^{\varepsilon,x}_t)_{t\ge 0}$ of a fixed bounded domain $D \subset \mathbb{R}^d$ under **Condition 1.1.1-(ii)** as $\varepsilon \to 0$. This study is dependent of the form of the underlying measure ν of the stochastic perturbation that is considered using adequate large/moderate deviations estimates. From now on we consider the jump measure ν given as

$$\nu(dz) = e^{-|z|^\alpha} dz, \tag{1.1.5}$$

for some $\alpha > 0$, where dz is the Lebesgue measure in the Borel measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. This measure is an exponentially light measure and it is a benchmark in the class of the Lévy measures since it exhibits the lightness of the jumps of the respective Lévy processes in terms of the parameter $\alpha > 0$ in comparison with the heavy-tailed jump measures that capture respectively the occurrence of big jumps in terms of a heaviness parameter.

Remark 1.1.3. *If $\alpha \geq 1$ the measure ν belongs to the class of **superexponential light jump measures** and $(X_t^{\varepsilon,x})_{t\geq 0}$ that solves (1.1.4) in the sense of **Definition 1.1.1** is called a **superexponential light jump process**. If $\alpha \in (0,1)$ ν is a **subexponential light jump measure** and $(X_t^{\varepsilon,x})_{t\geq 0}$ that solves (1.1.4) is called a **subexponential light jump process**.*

Remark 1.1.4 (Exponential integrability of ν). *We remark that the definition of ν yields the following:*

$$\int_{\mathbb{R}^d} e^{\sigma |z|^\beta} \nu(dz) < \infty \quad \text{for all } \sigma \geq 0, \text{ and } 0 \leq \beta < \alpha. \tag{1.1.6}$$

Remark 1.1.5 (Finite intensity $\nu(\mathbb{R}^d) < \infty$). *The intensity of the measure ν is finite, more specifically,*

$$\nu(\mathbb{R}^d) = \frac{c_d}{\alpha} \Gamma\Big(\frac{d}{\alpha}\Big),$$

for some constant $c_d > 0$, which depends on the dimension d and where Γ is Euler's Γ-function defined as

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx, \quad s \in \mathbb{R}. \tag{1.1.7}$$

This fact is a fundamental characteristic of the Lévy-driven systems given by (1.1.4) that we study in this thesis and it has deep consequences in the sequel of this work. For a proof we refer the reader to **Section A.3** of the **Appendix**.

Definition 1.1.2 (Entropy functional). *For every $T > 0$ we consider the entropy functional,*

$$\mathfrak{L}_T(g) := \int_0^T \int_{\mathbb{R}^d} (g(s,z) \ln g(s,z) - g(s,z) + 1) \nu(dz) ds, \tag{1.1.8}$$

defined for every measurable function $g : [0,T] \times \mathbb{R}^d \longrightarrow [0,\infty)$.

For every $T > 0$ and $M \geq 0$, we define

$$S^M := \Big\{ g : [0,T] \times \mathbb{R}^d \longrightarrow [0,\infty) \text{ measurable } \mid \mathfrak{L}_T(g) \leq M \Big\}. \tag{1.1.9}$$

For every $T > 0$, $M \geq 0$ and $g \in S^M$ we associate the measure

$$\nu_T^g(A) := \int_A g(s,z)\nu(dz)ds \quad \text{ for all } A \in \mathcal{B}([0,T] \times \mathbb{R}^d).$$

We denote by

$$\mathbb{S} := \bigcup_{M \geq 0} S^M.$$

Proposition 1.1.2. *Given $g \in \mathbb{S}$, $x \in \mathbb{R}^d$, and fixed $T \geq 0$, there exists a unique solution $\tilde{X}^g \in C([0,T], \mathbb{R}^d)$ of the equation*

$$\tilde{X}_t^g = x - \int_0^t \nabla U(\tilde{X}_s^g)\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} z(g(s,z) - 1)\nu(\mathrm{d}z)\mathrm{d}s, \text{ for all } t \in [0,T]. \tag{1.1.10}$$

For all $M \geq 0$, $x \in \mathbb{R}^d$ and $T \geq 0$, the solution of (1.1.10) satisfies the uniform bound

$$\sup_{0 \leq t \leq T} \sup_{g \in S^M} |\tilde{X}_t^g| < \infty. \tag{1.1.11}$$

Hence, the map

$$\begin{aligned} \mathcal{G}^0 : \mathbb{S} \longrightarrow & C([0,T], \mathbb{R}^d) \subset \mathbb{D}([0,T], \mathbb{R}^d), \\ \mathcal{G}^0(g) := & \tilde{X}^g, \quad g \in \mathbb{S}, \end{aligned}$$

is well defined.

A proof can be found in **section A.1.2** of the Appendix.

1.2 The superexponential regime $\alpha \geq 1$.

For every $\varepsilon > 0$, we consider the case that $\tilde{N}^{\frac{1}{\varepsilon}}$ is a compensated Poisson random measure defined on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ with compensator $\frac{1}{\varepsilon} ds \otimes \nu$, where

$$\nu(dz) = e^{-|z|^\alpha} dz, \quad \text{for some } \alpha \geq 1.$$

For every $\varepsilon > 0$, $T > 0$ and $x \in \mathbb{R}^d$, we state a large deviations principle for $(X^{\varepsilon,x})_{0 \leq t \leq T}$ that solves (1.1.4) in the sense of **Definition 1.1.1**.
We fix a bounded domain $D \subset \mathbb{R}^d$ under **Condition 1.1.1** such that $0 \in D$ and $x \in D$. It is our object of study the asymptotic behaviour of

$$\sigma^\varepsilon(x) := \inf\{t \geq 0 \mid X_t^{\varepsilon,x} \notin D\}, \quad \text{as } \varepsilon \to 0. \tag{1.2.1}$$

For every $T > 0$ and $\varphi \in \mathbb{D}([0,T], \mathbb{R}^d)$ we write the pre-image

$$\mathbb{S}_\varphi := \{g \in \mathbb{S} : \varphi = \mathcal{G}^0(g)\}.$$

Let

$$\mathbb{J} : \mathbb{D}([0,T], \mathbb{R}^d) \longrightarrow [0, \infty]$$

be defined by

$$\mathbb{J}(\varphi) := \inf_{g \in \mathbb{S}_\varphi} \int_0^T \int_{\mathbb{R}^d} (g(s,z) \ln g(s,z) - g(s,z) + 1) \nu(dz) ds. \tag{1.2.2}$$

For every $T > 0$ we define the set Λ of the increasing homeomorphisms $\lambda : [0,T] \longrightarrow [0,T]$.
We equip the space $\mathbb{D}([0,T], \mathbb{R}^d)$ with the topology generated by the metric defined by

$$d_{J_1}(\varphi, \psi) := \inf_{\lambda \in \Lambda} \Big(\sup_{t \in [0,T]} |\lambda(t) - t| + \sup_{t \in [0,T]} |\varphi(\lambda(t)) - \psi(t)| \Big), \tag{1.2.3}$$

for all $\varphi, \psi \in \mathbb{D}([0,T], \mathbb{R}^d)$. When equiped with the topology generated by the J_1-metric d_{J_1} the space $\mathbb{D}([0,T], \mathbb{R}^d)$ is called the *Skorokhod space*. For more details we refer the reader to **Section C.2.** of the Appendix.

Theorem 1.2.1 (**A large deviations principle in the superexponential regime**). *Let $T > 0$, **Condition 1.1.1**, **Condition 1.1.2** and ν defined in (1.1.5) for some $\alpha \geq 1$ be satisfied. Then, for all $x \in \mathbb{R}^d$, the family $(X^{\varepsilon,x})_{\varepsilon > 0}$ of stochastic processes that solve (1.1.4) in the sense of **Definition 1.1.1** satisfies a large deviations principle with good rate function $\mathbb{J}$ in the Skorokhod space $\mathbb{D}([0,T], \mathbb{R}^d)$.*

For every $T \geq 0$, $\Phi \in \mathbb{D}([0,T], \mathbb{R}^d)$, $t \in [0,T]$ and $x \in \mathbb{R}^d$ we define

$$\begin{aligned} \mathbb{J}(\Phi)_{x,t} := \inf_g \Big\{ &\int_0^t \int_{\mathbb{R}^d} (g(s,z) \ln g(s,z) - g(s,z) + 1) \nu(dz) ds : g \in \mathbb{S} \text{ such that} \\ &\Phi(s) = x - \int_0^s \nabla U(\Phi(r)) \mathrm{d}r + \int_0^s \int_{\mathbb{R}^d} z(g(r,z) - 1) \nu(\mathrm{d}z) \mathrm{d}r \ , 0 \leq s \leq t \Big\}. \end{aligned}$$

Furthermore, given $T \geq 0$, we define the cost function,

$$V(x,z,t) := \inf_{\Phi \in \mathbb{D}([0,T],\mathbb{R}^d):\Phi(t)=z} \mathbb{J}(\Phi)_{x,t}, \quad \text{for all } x,z \in \mathbb{R}^d, t \in [0,T].$$

We define also

$$V(x,z) = \inf_{t>0} V(x,z,t), \ \ x,z \in \mathbb{R}^d.$$

We call $V(0,z)$ the quasi-potential of (1.1.4).
Fix $D \subset \mathbb{R}^d$ under **Condition 1.1.1**. We define the potential as

$$\bar{V} := \inf_{z \notin D} V(0,z). \tag{1.2.4}$$

In **Proposition 2.3.1**, under our assumptions, we show that $\bar{V} < \infty$.

Remark 1.2.1. *The study of the first exist time for Brownian diffusions in terms of a potential is called in the literature as The Freidlin-Wentzell theory. We refer the reader to Freidlin and Wentzell (1988).*

Theorem 1.2.2 (The first exit time in the superexponential regime). *We assume **Condition 1.1.1**, **Condition 1.1.2** and ν given by (1.1.5) for some $\alpha \geq 1$. Then for any $\delta > 0$ and $x \in D$, we have*

$$\lim_{\varepsilon \to 0} \bar{\mathbb{P}}\Big(e^{\frac{\bar{V}-\delta}{\varepsilon}} < \sigma^\varepsilon(x) < e^{\frac{\bar{V}+\delta}{\varepsilon}}\Big) = 1.$$

Furthermore, for all $x \in D$, we have

$$\lim_{\varepsilon \to 0} \varepsilon \ln \bar{\mathbb{E}}[\sigma^\varepsilon(x)] = \bar{V}.$$

1.3 The subexponential regime $\alpha \in (0,1)$.

We assume now that, for every $\varepsilon > 0$, $\tilde{N}^{\frac{1}{\varepsilon}}$ is a compensated Poisson random measure defined on the probability space $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ with compensator $\frac{1}{\varepsilon} ds \otimes dz$, where the jump measure ν is defined by (1.1.5) for some $\alpha \in (0,1)$.
For every $T > 0$ we denote the space $L^2(\nu_T)$ of the square integrable functions defined on $[0,T] \times \mathbb{R}^d$ with values in $\mathbb{R}$ with respect to the measure $\nu_T := ds \otimes \nu$.
Let us fix a measurable function $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that we have, as $\varepsilon \to 0$,

$$a(\varepsilon) \to 0 \quad \text{and}$$
$$b(\varepsilon) := \frac{\varepsilon}{a^2(\varepsilon)} \to 0.$$

A moderate deviations principle and asymptotics of exit. For every $x \in \mathbb{R}^d$, we state a moderate deviations principle for the family $(X^{\varepsilon,x})_{\varepsilon>0}$ with speed $b(\varepsilon) = \varepsilon^{\alpha}$. The good rate function that is associated to this moderate deviations principle is a trivial function that assigns two values: 0 for the underlying deterministic dynamical system $(X^{0,x}_t)_{t\geq 0}$ and ∞ otherwise. For this reason, since the trajectories of the dynamical system described by (1.1.1) are attracted to $0 \in \mathbb{R}^d$, fixed a bounded domain $D \subset \mathbb{R}^d$ under **Condition 1.1.1** the potential associated to D only assigns the value ∞. This is not the right setting to describe the asymptotics of the law of $\sigma^{\varepsilon}(x)$, $x \in D$ in terms of such potential as $\varepsilon \to 0$ and it can be seen as a consequence of the difference of scales between the speed of convergence ε^{α} of the moderate deviations principle and the intensity parameter $\varepsilon > 0$ of the stochastic perturbation $(\tilde{L}^{\varepsilon}_t)_{t\geq 0}$ described in **Condition (1.1.2)**. Therefore, we state a moderate deviation principle for the renormalized deviation process defined by,

$$Y^{\varepsilon,x} := \frac{X^{\varepsilon,x} - X^{0,x}}{a(\varepsilon)}, \tag{1.3.1}$$

with $a(\varepsilon) = \varepsilon^{\frac{1-\alpha}{2}}$, for every $\varepsilon > 0$ and $x \in \mathbb{R}^d$. Secondly we state the first exit time problem for the exit of $(Y^{\varepsilon,x})_{t\geq 0}$ from a bounded domain $D \subset \mathbb{R}^d$ under **Condition 1.1.1**. We refer the reader to *Keblaner and Lipster (1999)* for a moderate deviations principle for a stochastic perturbation of a discrete dynamical system and the study of the first exit time of the respective renormalized process in the case of periodic dynamics.

Theorem 1.3.1. *Let $T > 0$, **Condition 1.1.1**, **Condition 1.1.2** be satisfied and ν defined in (1.1.5) for some $\alpha \in (0,1)$. Then for all $x \in \mathbb{R}^d$ the family $(X^{\varepsilon,x})_{\varepsilon>0}$ satisfies a large deviations principle with speed ε^{α} in the Skorokhod space $\mathbb{D}([0,T],\mathbb{R}^d)$ and with good rate function*

$$\tilde{\mathbb{I}}_0 : \mathbb{D}([0,T],\mathbb{R}^d) \longrightarrow [0,\infty],$$
$$\tilde{\mathbb{I}}_0(\eta) = \begin{cases} 0 & \textit{if } \eta = X^0 \\ \infty & \textit{otherwise}, \end{cases} \tag{1.3.2}$$

where $(X_t^{0,x})_{t\in[0,T]}$ *is the unique continuous solution of*

$$X_t^{0,x} = x - \int_0^t \nabla U(X_s^{0,x})ds, \quad t \in [0,T].$$

For every $\varepsilon > 0$, let us fix a bounded domain $D \subset \mathbb{R}^d$ under **Condition 1.1.1** and $x \in D$. Let $T \geq 0$, $\Phi \in \mathbb{D}([0,T],\mathbb{R}^d)$, $t \in [0,T]$ and we define

$$\tilde{\mathbb{I}}_0(\Phi)_{x,t} = \begin{cases} 0, & \text{if } \Phi(s) = x - \int_0^s \nabla U(\Phi(r))dr, \quad s \in [0,t], \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, given $T \geq 0$, we define the cost function,

$$V_0(x,z,t) := \inf_{\Phi\in\mathbb{D}([0,T],\mathbb{R}^d):\Phi(t)=z} \tilde{\mathbb{I}}_0(\Phi)_{x,t}, \quad \text{for all } x,z \in \mathbb{R}^d, t \in [0,T].$$

Due to the way $\tilde{\mathbb{I}}_0$ is defined we conclude that

$$V_0(x,z,t) = \begin{cases} 0 & \text{if } z = X_t^{0,x} \\ \infty & \text{otherwise.} \end{cases}$$

We define also

$$V_0(x,z) = \inf_{t>0} V_0(x,z,t), \;\; x,z \in \mathbb{R}^d.$$

We call $V_0(0,z)$ the quasi-potential and we define the potential as

$$\bar{V}_0 := \inf_{z\notin D} V_0(0,z).$$

Condition 1.1.1 implies that the image of X^0 is contained on D. Therefore,

$$\bar{V}_0 = \infty.$$

For this reason, for every $x \in \mathbb{R}^d$, we derive moderate deviations principle for $(Y^{\varepsilon,x})_{\varepsilon>0}$ and study the law and the expected value of the first exit time of the renormalized deviation process $(Y^\varepsilon)_{\varepsilon>0}$, defined by (1.3.1).
For every $x \in \mathbb{R}^d$, let us define

$$\tilde{\mathbb{I}}_1 : \mathbb{D}([0,T],\mathbb{R}^d) \longrightarrow [0,\infty],$$

$$\tilde{\mathbb{I}}_1(\eta) = \inf_{\psi\in\mathbb{T}_\eta} \frac{1}{2}||\psi||^2_{L^2(\nu_T)},$$

where

$$\mathbb{T}_\eta := \Big\{\psi \in L^2(\nu_T) \mid \eta(t) = -\int_0^t \nabla^2 U(X_s^0)\eta(s)ds + \int_0^t \int_{\mathbb{R}^d} z\psi(s,z)\nu(dz)ds, \quad 0 \leq t \leq T\Big\}.$$

Theorem 1.3.2 (**A moderate deviations principle in the subexponential regime for the renormalized deviation**). *Let $T > 0$, **Condition 1.1.1**, **Condition 1.1.2** be satisfied and ν defined in (1.1.5) for some $\alpha \in (0, 1)$. Then, for all $x \in \mathbb{R}^d$, the family $(Y^{\varepsilon,x})_{\varepsilon>0}$ defined in (1.3.1) satisfies a large deviations principle with speed ε^α and with good rate function $\tilde{\mathbb{I}}_1$ in the Skorokhod space $\mathbb{D}([0, T], \mathbb{R}^d)$.*

Remark 1.3.1. *Let us consider $\eta \in \mathbb{D}([0, T], \mathbb{R}^d)$ and $\psi \in L^2(\nu_T)$ such that*

$$\eta(t) = -\int_0^t \nabla^2 U(X_s^0)\eta(s)ds + \int_0^t \int_{\mathbb{R}^d} z\psi(s, z)\nu(dz)ds, \quad 0 \le t \le T.$$

*This controlled ODE that appears in the definition of $\tilde{\mathbb{I}}_1$ has a different structure than the controlled ODE that is part of the definition of the good rate function $\mathbb{J}$ described in (1.2.2). This will be clear in the proof of **Theorem 1.3.2**. We just say some brief words to motivate the reader about the role of $-\nabla^2 U(X_s^{0,x})$ in this controlled ODE. This is related with the trivial identity $X_s^{\varepsilon,x} = X_s^{0,x} + a(\varepsilon)Y_s^{\varepsilon,x}$ and with the Taylor development $-\nabla U(X_s^{\varepsilon,x}) + \nabla U(X_s^{0,x}) = -a(\varepsilon)\nabla^2 U(X_s^{0,x})Y_s^\varepsilon + R_s^\varepsilon$, for some rest R_s^ε which properties are specified during the proof of **Theorem 1.3.2**.*

For given $x \in D$, we are interested in the asymptotic study of the law and the expected value of

$$\tilde{\sigma}^\varepsilon(x) := \inf\{t \ge 0 \mid Y_t^{\varepsilon,x} \notin D\}, \quad \text{as } \varepsilon \to 0, \tag{1.3.3}$$

which is the first exit time of $X^{\varepsilon,x}$ from the translated domain $X^{0,x} + a(\varepsilon)D$, as $\varepsilon \to 0$.

For every $T \ge 0$, $\eta \in \mathbb{D}([0, T], \mathbb{R}^d)$, $t \in [0, T]$ and $x \in \mathbb{R}^d$, we define

$$\begin{aligned}\tilde{\mathbb{I}}_1(\eta)_{x,t} = \inf \Big\{ &\frac{1}{2}\int_0^t \int_{\mathbb{R}^d} |\psi(s, z)|^2 \nu(dz)ds \mid \psi \in L^2(\nu_T) \text{ such that} \\ &\eta(t) = x - \int_0^t \nabla^2 U(X_s^0)\eta(s)ds + \int_0^t \int_{\mathbb{R}^d} z\psi(s, z)\nu(dz)ds, \quad 0 \le t \le T \Big\},\end{aligned}$$

and the associated cost function,

$$V_1(x, z, t) := \inf_{\eta \in \mathbb{D}([0,t],\mathbb{R}^d):\eta(t)=z} \tilde{\mathbb{I}}_1(\eta)_{x,t}, \text{ for all } x, z \in \mathbb{R}^d, \quad t \in [0, T],$$

We define

$$V_1(x, z) := \inf_{t>0} V_1(x, z, t) \quad x, z \in \mathbb{R}^d,$$

and we call $V(0, z)$ the quasi-potential. We define the potential as

$$\bar{V}_1 := \inf_{z \notin D} V_1(0, z). \tag{1.3.4}$$

It is shown in **Proposition 3.3.1**. that $\bar{V}_1 < \infty$.

Theorem 1.3.3 (The first exit time in the subexponential regime). *We assume* ***Condition 1.1.1****,* ***Condition 1.1.2****, and ν given by (1.1.5) for some $\alpha \in (0,1)$. Then for any $\delta > 0$ and $x \in D$, we have*

$$\lim_{\varepsilon \to 0} \bar{\mathbb{P}}\Big(e^{\frac{\bar{V}_1 - \delta}{\varepsilon^\alpha}} < \tilde{\sigma}^\varepsilon(x) < e^{\frac{\bar{V}_1 + \delta}{\varepsilon^\alpha}}\Big) = 1.$$

Furthermore, for all $x \in D$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^\alpha \ln \bar{\mathbb{E}}[\tilde{\sigma}^\varepsilon(x)] = \bar{V}_1.$$

Chapter 2

A large deviations principle and the first exit time asymptotics for superexponential jump diffusions

2.1 A sufficient condition for a large deviations principle

In this chapter we assume the setup discussed in the first section of **Chapter 1**, in particular **Condition (1.1.1)**, **Condition (1.1.2)**, with ν given by (1.1.5) for $\alpha \geq 1$. Fix $T > 0$. For every $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we consider the solution $(X_t^{\varepsilon,x})_{t\in[0,T]}$ of (1.1.4).
We state a sufficient condition, obtained in *Budhiraja et al. (2011)* for a large deviations principle of the laws of the family $(X^{\varepsilon,x})_{\varepsilon>0}$. This allows to determine the asymptotics when $\varepsilon \to 0$ of the law of the first exit time of $(X^{\varepsilon,x})_{t\geq 0}$ from the bounded domain D.
We denote by $\bar{\mathcal{P}}$ the predictable σ-field on $[0,T]\times\bar{\mathbb{M}}$ with respect to the filtration $(\bar{\mathcal{F}}_t)_{0\leq t\leq T}$, that is the σ-algebra generated on $[0,T]\times\bar{\mathbb{M}}$ by all $(\bar{\mathcal{F}}_t)_{0\leq t\leq T}$-adapted càdlàg processes. We define the space of positive controls

$$\bar{\mathcal{A}}^+ := \Big\{\varphi : [0,T]\times\mathbb{R}^d\times\bar{\mathbb{M}} \longrightarrow [0,\infty) \mid \varphi \text{ is } (\bar{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R}^d),\mathcal{B}([0,\infty))) \text{ measurable } \Big\}.$$

We consider a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ and, for every $n \in \mathbb{N}$, we define the set of the n-bounded positive controls

$$\begin{aligned}\bar{\mathcal{A}}^+_{b,n} := \{\varphi \in \bar{\mathcal{A}}^+ \mid & \text{ for all } (t,\bar{m}) \in [0,T]\times\bar{\mathbb{M}} :\\ & \frac{1}{n} \leq \varphi(t,x,\bar{m}) \leq n, \text{ if } x \in K_n, \text{ and } \quad \varphi(t,x,\bar{m}) = 1, \text{ if } x \in K_n^c\}.\end{aligned}$$

Furthermore, set

$$\bar{\mathcal{A}}^+_b := \bigcup_{n\in\mathbb{N}} \bar{\mathcal{A}}^+_{b,n}$$

and for every $M \geq 0$,

$$\mathcal{U}^M_+ := \{u \in \bar{\mathcal{A}}^+_b : u(.,.,\bar{m}) \in S^M \quad \bar{\mathbb{P}} - \text{a.s }, \text{ for } S^M \text{ defined in (1.1.9)}\}.$$

For every $g \in S^M$ we associate the measure

$$\nu_T^g(A) := \int_A g(s,z)\nu(dz)ds \quad \text{for all } A \in \mathcal{B}([0,T] \times \mathbb{R}^d).$$

We identify S^M with the space of the measures $\{\nu_T^g \mid g \in S^M\} \subset \mathbb{M}$ equiped with the topology induced by the weak convergence on the compact sets of $\mathbb{M}$. We refer the reader to **Section D.5.1** and **Section D.5.3** of the Appendix. Essentially this convergence is equivalent to the convergence in the vague topology (see **Definition B.2.2**). **Proposition D.5.1** ensures that this identification produces a topology in S^M under which S^M is compact.

In what follows, we state the sufficient condition obtained in *Budhiraja et al. (2011)* for the large deviations principle.

Condition 2.1.1 (**LDP condition**). *Let $\mathcal{D}$ be a Polish space and $(\mathcal{G}^\varepsilon)_{\varepsilon>0}$ be a family of measurable maps $\mathcal{G}^\varepsilon : \mathbb{M} \longrightarrow \mathcal{D}$ and $\mathcal{G}^0 : \{\nu_T^g \mid g \in \mathbb{S}\} \longrightarrow \mathcal{D}$ a measurable map such that the following conditions hold.*

(i) ***Continuity in the control for the underlying deterministic system.*** *For every $M \geq 0$ and $n \in \mathbb{N}$, let $g_n, g \in S^M$ such that $\nu_T^{g_n} \to \nu_T^g$ weakly on the compact sets of $\mathbb{M}$. Then there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}} \subset (g_n)_{n\in\mathbb{N}}$ such that*

$$\mathcal{G}^0(\nu_T^{g_{n_k}}) \to \mathcal{G}^0(\nu_T^g), \quad \text{as } k \to \infty,$$

for the topology induced by the metric in $\mathcal{D}$.

(ii) ***Weak law of large numbers for the controlled stochastic systems.*** *For every $M \geq 0$ and $\varepsilon > 0$, let $\varphi_\varepsilon, \varphi \in \mathcal{U}_+^M$ such that we have the convergence in law $\varphi_\varepsilon \Rightarrow \varphi$, as $\varepsilon \to 0$. Then $\mathcal{G}^0(\nu_T^\varphi)$ is a limit point in law of $\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon})$ as $\varepsilon \to 0$.*

Let $\mathcal{D}$ be a Polish space and $\mathcal{G}^0 : \mathbb{M} \longrightarrow \mathcal{D}$. For $\varphi \in \mathcal{D}$ we define

$$\mathbb{S}_\varphi := \{g \in \mathbb{S} : \varphi = \mathcal{G}^0(\nu_T^g)\}$$

and

$$\begin{aligned} &\mathbb{J} : \mathcal{D} \longrightarrow [0,\infty], \\ &\mathbb{J}(\varphi) := \inf_{g\in\mathbb{S}_\varphi} \int_0^T \int_{\mathbb{R}^d} (g(s,z)\ln g(s,z) - g(s,z) + 1)\nu(\mathrm{d}z)\mathrm{d}s. \end{aligned} \tag{2.1.1}$$

The following theorem is proved in **Section D.6** of the Appendix.

Theorem 2.1.1. *Let $\mathcal{D}$ be a Polish space and $(\mathcal{G}^\varepsilon)_{\varepsilon>0}$ be a family of measurable maps $\mathcal{G}^\varepsilon : \mathbb{M} \longrightarrow \mathcal{D}$ and $\mathcal{G}^0 : \mathbb{M} \longrightarrow \mathcal{D}$ a measurable map satisfying **Condition 2.1.1**. Then $\mathbb{J}$ defined in (2.1.1) is a good rate function and $(Z^\varepsilon)_{\varepsilon>0}$ defined for all $\varepsilon > 0$ by $Z^\varepsilon := \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}})$ satisfies a large deviations principle in $\mathcal{D}$ with speed $b(\varepsilon) = \varepsilon$ and with good rate function $\mathbb{J}$.*

Remark 2.1.2. *Given $x \in \mathbb{R}^d$ and $T > 0$, due to* ***Theorem 2.1.1****, we use* ***Condition 2.1.1*** *to prove the large deviations principle stated in* ***Theorem 1.2.1*** *for $(X_t^{\varepsilon,x})_{t\in[0,T]}$, solution of (1.1.4) in the sense of* ***Definition 1.1.1****. In this framework, the measurable maps of* ***Condition 2.1.1*** *are defined, for every $\varepsilon > 0$ and $x \in \mathbb{R}^d$, by $\mathcal{G}^{\varepsilon,x}(\varepsilon N^{\frac{1}{\varepsilon}}) := X^{\varepsilon,x}$. We write $\mathcal{G}^{\varepsilon,x}$ to stress the dependence on the initial condition $x \in \mathbb{R}^d$ of $(X_t^{\varepsilon,x})_{t\in[0,T]}$. The map $\mathcal{G}^{0,x}$ is defined by*

$$\begin{aligned} &\mathcal{G}^{0,x} : \{\nu_T^g \mid g \in \mathbb{S}\} \longrightarrow C([0,T], \mathbb{R}^d), \\ &\mathcal{G}^{0,x} := \tilde{X}^g, \quad g \in \mathbb{S} \end{aligned}$$

where $\tilde{X}^g$ is the unique continuous solution of the controlled ODE (1.1.10).

2.2 A large deviations principle

Preparations

We start with a technical lemma that will be crucial in the proof of **Theorem 1.2.1**.

Lemma 2.2.1. *Assume $\nu(dz) = e^{-|z|^\alpha}dz$ for some $\alpha > 0$. Fix a measurable function $G : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ which is locally bounded in the first variable x and with polynomial growth in the second variable $z \in \mathbb{R}^d$. Then for every $x \in \mathbb{R}^d$ and $M > 0$ we have the following statements.*

1.

$$\sup_{g\in S^M} \int_{[0,T]\times\mathbb{R}^d} |G(x,z)|g(s,z)\nu(dz)ds < \infty, \tag{2.2.1}$$

2.

$$\sup_{g\in S^M} \int_{[0,T]\times\mathbb{R}^d} |G(x,z)|^2 g(s,z)\nu(dz)ds < \infty, \tag{2.2.2}$$

3.

$$\lim_{\delta\to 0} \sup_{g\in S^M} \sup_{\substack{0\le s<t \\ |t-s|\le\delta}} \int_{[s,t]\times\mathbb{R}^d} |G(x,z)||g(s,z)-1|\nu(dz)ds = 0. \tag{2.2.3}$$

Remark 2.2.1.

*i) The measure ν is assumed to be only exponentially light, covering both superexponential and subexponential regimes. The preceding lemma will be also used in **Chapter 4** to prove **Theorem 4.5.1**.*

*ii) The function $G(x,z) := z, (x,z) \in \mathbb{R}^d$, satisfies the assumptions of the last lemma. Therefore, we use it in the sequel to prove the large deviations principle stated in **Theorem 1.2.1** for $(X_t^{\varepsilon,x})_{t\in[0,T]}$.*

*Proof of **Lemma 2.2.1**.*

1. We start with the proof of (2.2.1). Given $x \in \mathbb{R}^d$ and $R = |x|$, there exist $r \in \mathbb{N}$ and $K = K(x) > 0$ such that

$$|G(x,z)| \le K(1+|z|^r) \quad \text{for all } z \in \mathbb{R}^d. \tag{2.2.4}$$

This implies for any fixed $g \in S^M$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |G(x,z)|g(s,z)\nu(dz)ds \\
&\le K \int_{[0,T]\times\mathbb{R}^d} g(s,z)\nu(dz)ds + K \int_{[0,T]\times\mathbb{R}^d} |z|^r g(s,z)\nu(dz)ds.
\end{aligned}$$

Using *Young's inequality* (D.7.1) for the entropy function $\ell(b) = b\ln b - b + 1$, $b \geq 0$, observing that $\nu(\mathbb{R}^d) < \infty$ (**Remark 1.1.5**) and $g \in S^M$, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |G(x,z)| g(s,z) \nu(dz, ds) \\
&\leq 2eK\nu(\mathbb{R}^d)T + K\int_0^T \int_{\mathbb{R}^d} \ell(g(s,z))\nu(dz)ds + K\int_0^T \int_{\mathbb{R}^d} \ell(|z|^r g(s,z))\nu(dz)ds \\
&\leq 2eK\nu(\mathbb{R}^d)T + KM + K\int_0^T \int_{\mathbb{R}^d} \ell(|z|^r g(s,z))\nu(dz)ds. \qquad (2.2.5)
\end{aligned}$$

We define the measurable set

$$E := \{(s,z) \in [0,T] \times \mathbb{R}^d \mid |z|^r g(s,z) \geq 1\}$$

and divide the remaining term

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}^d} \ell(|z|^r g(s,z))\nu(dz)ds \\
&= \int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(|z|^r g(s,z))\nu(dz)ds + \int_{([0,T]\times\mathbb{R}^d)\cap E^c} \ell(|z|^r g(s,z))\nu(dz)ds. \qquad (2.2.6)
\end{aligned}$$

On E^c we have $|z|^r g(s,z) < 1$ which implies $\ell(|z|^r g(s,v)) \leq 1$. Therefore,

$$\int_{([0,T]\times\mathbb{R}^d)\cap E^c} \ell(|z|^r g(s,z))\nu(dz)ds \leq \nu(\mathbb{R}^d)T < \infty. \qquad (2.2.7)$$

On E we have $|z|^r g(s,z) \geq 1$. *Young's L^p inequality* (**Remark A.1.2** in the Appendix), the monotonicity and the convexity of ℓ in $[1,+\infty)$ yield, for any conjugate exponents $p, q \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\ell(|z|^r g(s,z)) \leq \ell\Big(\frac{1}{p}|z|^{rp} + \frac{1}{q}(g(s,z))^q\Big) \leq \frac{1}{p}\ell(|z|^{rp}) + \frac{1}{q}\ell((g(s,z))^q).$$

Consequently,

$$\begin{aligned}
&\int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(|z|^r g(s,z))\nu(dz)ds \\
&\leq \frac{1}{p}\int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(|z|^{rp})\nu(dz)ds + \int_{([0,T]\times\mathbb{R}^d)\cap E} \frac{1}{q}\ell(g(s,z)^q)\nu(dz)ds. \qquad (2.2.8)
\end{aligned}$$

Due to *Fatou's lemma*,

$$\begin{aligned}
\limsup_{q\to 1+} \int_{([0,T]\times\mathbb{R}^d)\cap E} \frac{1}{q}\ell(g(s,z)^q)\nu(dz)ds &\leq \int_{([0,T]\times\mathbb{R}^d)\cap E} \limsup_{q\to 1+} \frac{1}{q}\ell(g(s,z)^q)\nu(dz)ds \\
&= \int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(g(s,z))\nu(dz)ds \\
&\leq M.
\end{aligned}$$

This implies the existence of $q_0 > 1$ such that

$$\frac{1}{q_0}\int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(g(s,z)^{q_0})\nu(dz)ds \leq M.$$

Hence the corresponding convex conjugate is $p_0 := \dfrac{q_0}{q_0-1}$ and

$$\begin{aligned}
&\int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(|z|^r g(s,z))\nu(dz)ds \\
&\leq \frac{1}{p_0}\int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(|z|^{rp_0})\nu(dz)ds + \frac{1}{q_0}\int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(|g(s,z)|^{q_0})\nu(dz)ds \\
&\leq \int_{([0,T]\times\mathbb{R}^d)\cap E} \ell(|z|^{rp_0})\nu(dz)ds + M. \qquad (2.2.9)
\end{aligned}$$

There exists $R > 0$ such that $\ell(|z|^{rp_0}) \leq |z|^{rp_0+1}$ on $\{|z| \geq R\}$. Therefore

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}^d} \ell(|z|^{rp_0})\nu(dz)ds \\
&\leq \int_{[0,T]\times\{|z|\geq R\}} |z|^{rp_0+1}\nu(dz)ds + \int_{[0,T]\times\{|z|<R\}} \ell(|z|^{rp_0})\nu(dz)ds. \qquad (2.2.10)
\end{aligned}$$

Since ℓ is bounded on $\{|z| \leq R\}$ and $\nu(\mathbb{R}^d) < \infty$ the second integral is finite. Using first the generalized spherical change of coordinates in $\mathbb{R}^d$ and after that the change of variables $t = |z|^\alpha$ we have

$$\begin{aligned}
\int_{[0,T]\times\mathbb{R}^d} |z|^{rp_0+1}e^{-|z|^\alpha}dzds &\leq 2T\pi^{d-1}\int_0^\infty t^{rp_0+d}e^{-t^\alpha}dt \\
&\leq \frac{2T\pi^{d-1}}{\alpha}\Gamma\Big(\frac{1+rp_0+d}{\alpha}\Big) \\
&< \infty, \qquad (2.2.11)
\end{aligned}$$

where Γ is *Euler's* Γ*- function*, defined in (1.1.7). Collecting (2.2.5), (2.2.6), (2.2.7), (2.2.8), (2.2.9), (2.2.10) and (2.2.11) the result follows.

2. Statement 2 follows with analogous arguments of statement 1.

3. We fix $M > 0$, $x \in \mathbb{R}^d$, $g \in S^M$ and $\delta' > 0$. Due to (2.2.4) and *Young's inequality*

(D.7.1), there exist $K = K(x) > 0$ and $\sigma > 0$ such that, for all $0 \le s \le t \le T$,

$$\begin{aligned}
&\int_s^t \int_{\mathbb{R}^d} |G(x,z)||g(s,z)-1|\nu(dz)ds \\
&\le K\Big(\int_s^t \int_{\mathbb{R}^d} g(s,z)\nu(dz)ds + \nu(\mathbb{R}^d)|t-s| \\
&+ \int_s^t \int_{\mathbb{R}^d} |z|^r g(s,z)\nu(dz)ds + \int_s^t \int_{\mathbb{R}^d} |z|^r \nu(dz)ds\Big) \\
&\le K \int_s^t \int_{\mathbb{R}^d} g(s,z)\nu(dz)ds + K(c_\nu(r) + \nu(\mathbb{R}^d))|t-s| \\
&+ Ke^{\sigma}\nu(\mathbb{R}^d)|t-s| + \frac{K}{\sigma}\int_s^t \int_{\mathbb{R}^d} \ell(|z|^r g(s,z))\nu(dz)ds, \qquad (2.2.12)
\end{aligned}$$

where $c = c_\nu(r) := \int_{\mathbb{R}^d} |z|^r \nu(dz) < \infty$, due to the exponential integrability property (1.1.6) of ν . Combining the statements (2.2.6), (2.2.7), (2.2.9), (2.2.10) and (2.2.11) we have

$$\int_0^T \int_{\mathbb{R}^d} \ell(|z|^r g(s,z))\nu(dz)ds < \infty.$$

We choose $\sigma > 0$ such that

$$\frac{K}{\sigma}\int_{[0,T]\times\mathbb{R}^d} \ell(|z|^r g(s,z))\nu(dz)ds < \frac{\delta'}{4}. \qquad (2.2.13)$$

Using *Young's inequality* (**Remark A.1.2**) and fixing $\sigma_1 > \frac{8KM}{\delta'}$ yields

$$\begin{aligned}
K\int_t^s \int_{\mathbb{R}^d} g(s,z)\nu(dz)ds &\le Ke^{\sigma_1}\nu(\mathbb{R}^d)|t-s| + \frac{K}{\sigma_1}\int_{[0,T]\times\mathbb{R}^d} \ell(g(s,z))\nu(dz)ds \\
&\le Ke^{\sigma_1}\nu(\mathbb{R}^d)|t-s| + \frac{MK}{\sigma_1} \\
&\le Ke^{\sigma_1}\nu(\mathbb{R}^d)|t-s| + \frac{\delta'}{8}. \qquad (2.2.14)
\end{aligned}$$

For any $\delta > 0$ satisfying

$$\delta < \frac{5\delta'}{8K(c_\nu + \nu(\mathbb{R}^d)(1+e^{\sigma}+e^{\sigma_1}))},$$

the estimates (2.2.12) and (2.2.14) imply, for any $0 \le s \le t \le T$ such that $|t-s| < \delta$,

$$\int_s^t \int_{\mathbb{R}^d} |G(x,z)||g(s,z)-1|\nu(dz)ds < \delta'.$$

This concludes the proof.

□

Proof of the large deviations principle stated in Theorem 1.2.1

We fix $\varepsilon > 0$, $T > 0$ and $x \in \mathbb{R}^d$. By **Theorem 1.1.1** there exists a measurable map

$$\mathcal{G}^{\varepsilon,x} : \mathbb{M} \longrightarrow \mathbb{D}([0,T], \mathbb{R}^d),$$

with respect to the topology given in $\mathbb{M}$ by weak convergence in the compact sets and to the J_1 topology in $\mathbb{D}([0,T], \mathbb{R}^d)$, such that $\mathcal{G}^{\varepsilon,x}(\varepsilon N^{\frac{1}{\varepsilon}}) := X^{\varepsilon,x}$ is the unique solution in the sense of **Definition 1.1.1** of (1.1.1).
Given $g \in \mathbb{S}$, **Lemma 1.1.2** implies the existence of a measurable map

$$\mathcal{G}^0 : \{\nu_T^g \mid g \in \mathbb{S}\} \longrightarrow C([0,T], \mathbb{R}^d)$$

such that $\mathcal{G}^{0,x}(\nu_T^g) := \tilde{X}^{g,x}$ is the unique continuous solution of (1.1.10).
Whenever possible without confusion of notation we omit the dependence on the initial condition $x \in \mathbb{R}^d$.

In order to prove **Theorem 1.2.1** we verify **Condition 2.1.1**. Fixed $M > 0$ and a family $(\varphi_\varepsilon)_{\varepsilon>0} \subset \mathcal{U}_+^M$, we set $\psi_\varepsilon = \frac{1}{\varphi_\varepsilon}$ for every $\varepsilon > 0$. The random measure $N^{\frac{1}{\varepsilon}\varphi_\varepsilon}$ is a controlled random measure, defined by,

$$N^{\frac{1}{\varepsilon}\varphi_\varepsilon}([0,t] \times U) := \int_0^t \int_U \int_0^\infty \mathbf{1}_{[0,\frac{1}{\varepsilon}\varphi_\varepsilon]} \bar{N}(ds, dx, dr) \quad \text{for all } t \in [0,T], U \in \mathcal{B}(\mathbb{R}^d).$$

Noting that $\varphi_\varepsilon \in \mathcal{U}_M^+$ means that φ_ε is bounded below and above on a certain compact of $[0,T] \times \mathbb{R}^d$ and $\varphi_\varepsilon = 1$ outside of that compact, it is immediate that ψ_ε satisfies the integrability condition of the version of *Girsanov's theorem* given in **Theorem B.3.2**. Therefore, the *Doleans-Dade exponential* of ψ_ε with respect to $\bar{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}$ under $\bar{\mathbb{P}}$, defined for $t \in [0,T]$ by

$$\mathcal{E}(\psi_\varepsilon)(t) := \\ \exp\Big(\int_0^t \int_{\mathbb{R}^d} \int_0^{\frac{1}{\varepsilon}} \ln \psi_\varepsilon(s,z) \bar{N}(ds, dz, dr) + \int_0^t \int_{\mathbb{R}^d} \int_0^{\frac{1}{\varepsilon}} (-\psi_\varepsilon(s,z) + 1)\nu(dz)drds\Big)$$

is an $(\bar{\mathcal{F}}_t)_{0\le t\le T}$ - martingale.
Girsanov's theorem (**Theorem B.3.2**) states that the measure defined as

$$\mathbb{Q}_T^\varepsilon(G) := \int_G \mathcal{E}(\psi_\varepsilon) d\bar{\mathbb{P}}(\bar{m}), \quad \text{for all } G \in \mathcal{B}(\bar{\mathbb{M}}),$$

is a probability measure on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$, the measures $\bar{\mathbb{P}}$ and $\mathbb{Q}_T^\varepsilon$ are mutually absolutely continuous and the controlled random measure $\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}$ under $\mathbb{Q}_T^\varepsilon$ has the same law as $\varepsilon N^{\frac{1}{\varepsilon}}$ under $\bar{\mathbb{P}}$ on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$. We do not stress the dependence of the integral with respect to $\mathbb{Q}_T^\varepsilon$.

We call by $\tilde{X}^{\varepsilon,x} := \mathcal{G}^{\varepsilon,x}(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon})$ the unique strong solution of the following controlled SDE, for every $t \in [0,T]$,

$$\tilde{X}_t^{\varepsilon,x} = x - \int_0^t \nabla U(\tilde{X}_s^\varepsilon) ds + \int_0^t \int_{\mathbb{R}^d} z(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds, dz) - \nu(dz)ds). \tag{2.2.15}$$

Proposition 2.2.1 (**A priori-estimates on the controlled processes** $(\tilde{X}_t^{\varepsilon,x})_{0\le t\le T}$**).** *For any fixed* $x \in \mathbb{R}^d$ *and* $T > 0$ *there exists* $\varepsilon_0 \in (0,1)$ *such that*

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\Big[\sup_{0\le s\le T} |\tilde{X}_s^{\varepsilon,x}|^2\Big] < \infty. \tag{2.2.16}$$

Proof. For convenience of notation we drop the dependence of $\tilde{X}^{\varepsilon,x}$ on $x \in \mathbb{R}^d$. Using *Ito's formula* (**Proposition B.3.2**) we have, for all $\varepsilon > 0$ and $t \in [0,T]$,

$$\begin{aligned}|\tilde{X}_t^\varepsilon|^2 &= |x|^2 + 2\int_0^t \langle -\nabla U(\tilde{X}_s^\varepsilon), \tilde{X}_s^\varepsilon\rangle ds + 2\int_0^t\int_{\mathbb{R}^d} \langle z, \tilde{X}_s^\varepsilon\rangle(\varphi_\varepsilon(s,z)-1)\nu(dz)ds \\ &+ \int_0^t\int_{\mathbb{R}^d} (|\varepsilon z|^2 + 2\langle \varepsilon z, \tilde{X}_{s-}^\varepsilon\rangle)(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz) - \frac{1}{\varepsilon}\varphi_\varepsilon(s,z)\nu(dz)ds) \\ &+ \varepsilon\int_0^t\int_{\mathbb{R}^d} |z|^2\varphi_\varepsilon(s,z)\nu(dz)ds. \end{aligned}\tag{2.2.17}$$

The second term in the preceeding sum is bounded, due to $\nabla U(0) = 0$ and the dissipativity of the potential $-\nabla U$ (1.1.2), as follows,

$$2\int_0^t \langle -\nabla U(\tilde{X}_s^\varepsilon), \tilde{X}_s^\varepsilon\rangle ds \le -2\eta\int_0^t |\tilde{X}_s^\varepsilon|^2 ds.$$

We treat the third term of (2.2.17) by

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{R}^d} \langle z, \tilde{X}_s^\varepsilon\rangle(\varphi_\varepsilon(s,z)-1)\nu(dz)ds\Big| \\ &\le \int_0^t\int_{\mathbb{R}^d} |z||\tilde{X}_s^\varepsilon||\varphi_\varepsilon(s,z)-1|\nu(dz)ds \\ &\le \int_0^t (1+2|\tilde{X}_s^\varepsilon|^2)\int_{\mathbb{R}^d} |z||\varphi_\varepsilon(s,z)-1|\nu(dz)ds \\ &\le C_1 + 2\int_0^t |\tilde{X}_s^\varepsilon|^2\Big(\int_{\mathbb{R}^d} |z||\varphi_\varepsilon(s,z)-1|\nu(dz)\Big)ds.\end{aligned}$$

Above we used in the last line the fact $a + a^2 \le 1 + 2a^2,$ for all $a > 0$, and

$$C_1 := \sup_{g\in S^M}\int_0^t\int_{\mathbb{R}^d} |z||g(s,z)-1|\nu(dz)ds < \infty$$

due to (2.2.1) of **Lemma 2.2.1**.

The last term of (2.2.17) can be estimated in the following way, due to (2.2.2) of **Lemma 2.2.1**,

$$\varepsilon\int_0^t\int_{\mathbb{R}^d} |z|^2\varphi_\varepsilon(s,z)\nu(dz)ds \le \varepsilon\sup_{g\in S^M}\int_0^t\int_{\mathbb{R}^d} |z|^2 g(s,z)\nu(dz)ds < \infty.$$

We now treat the remaining martingale part in (2.2.17), $M_t := M_t^1 + M_t^2$, for all $t \in [0,T]$, where

$$\begin{cases} M_t^1 := \int_0^t \int_{\mathbb{R}^d} |\varepsilon z|^2 \tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz), \\ M_t^2 := \int_0^t \int_{\mathbb{R}^d} 2\langle \varepsilon z, \tilde{X}_{s-}^{\varepsilon} \rangle \tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz). \end{cases}$$

Due to the integral version of *Gronwall's inequality* (**Proposition A.1.1**), there exists a constant $C_2 = C_2(T) > 0$ such that

$$\sup_{0\le s\le T} |\tilde{X}_s^\varepsilon|^2 \le C_2\Big(1 + \sup_{0\le s\le T} |M_s^1| + \sup_{0\le s\le T} |M_s^2|\Big). \tag{2.2.18}$$

Since the intensity ν satisfies $\nu(\mathbb{R}^d) < \infty$ (**Remark 1.1.5**) we decompose the compensated controlled random measure $\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}$

$$\begin{aligned} \bar{\mathbb{E}}\Big[\sup_{0\le s\le T} |M_s^1|\Big] &\le \bar{\mathbb{E}}\Big|\int_0^T \int_{\mathbb{R}^d} |\varepsilon z|^2 N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dsdz)\Big| \\ &\quad + \bar{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{R}^d} |\varepsilon z|^2 \frac{1}{\varepsilon}\varphi_\varepsilon(s,z)\nu(dz)ds\Big] \\ &\le 2\bar{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{R}^d} |\varepsilon z|^2 \frac{1}{\varepsilon}\varphi_\varepsilon(s,z)\nu(dz)ds\Big] \\ &\le \sup_{g\in S^M} 2\varepsilon\bar{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{R}^d} |z|^2 g(s,z)\nu(dz)ds\Big] \\ &\le 2\varepsilon C_3. \end{aligned} \tag{2.2.19}$$

Due to (2.2.2) the last expression is finite. The *Burkholder-Davis-Gundy inequality* (**Proposition B.3.3**), the fact $\sqrt{ab} \le \frac{a}{2} + \frac{b}{2}$ in the 5th line of the following estimate and

(2.2.2) imply that there exists some $C_4 > 0$ such that

$$\begin{aligned}
\bar{\mathbb{E}}\Big[\sup_{0\le s\le T}|M_s^2|\Big] &\le C_4\bar{\mathbb{E}}\Big[[M^2]_T^{1/2}\Big] \\
&\le C_4\bar{\mathbb{E}}\Big[\sqrt{\int_0^T\int_{\mathbb{R}^d}4\varepsilon^2|\langle z,\tilde{X}_{s-}^{\varepsilon}\rangle|^2 N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz)}\Big] \\
&\le 2C_4\varepsilon\bar{\mathbb{E}}\Big[\sqrt{\int_0^T\int_{\mathbb{R}^d}|z|^2|\tilde{X}_{s-}^{\varepsilon}|^2 N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz)}\Big] \\
&\le 2C_4\varepsilon\bar{\mathbb{E}}\Big[\sqrt{\sup_{0\le r\le T}|\tilde{X}_r^{\varepsilon}|^2\int_0^T\int_{\mathbb{R}^d}|z|^2 N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz)}\Big] \\
&\le C_4\varepsilon\Big(\bar{\mathbb{E}}\Big[\sup_{0\le s\le T}|\tilde{X}_s^{\varepsilon}|^2\Big]+\bar{\mathbb{E}}\Big[\int_0^T|z|^2 N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz)\Big]\Big) \\
&\le C_4\varepsilon\Big(\bar{\mathbb{E}}\Big[\sup_{0\le s\le T}|\tilde{X}_s^{\varepsilon}|^2\Big]+\bar{\mathbb{E}}\Big[\int_0^T\int_{\mathbb{R}^d}|z|^2\frac{1}{\varepsilon}\varphi_\varepsilon(s,z)\nu(dz)ds\Big]\Big) \\
&\le C_4\varepsilon\bar{\mathbb{E}}\Big[\sup_{0\le t\le T}|\tilde{X}_s^{\varepsilon}|^2\Big]+C_3. \qquad (2.2.20)
\end{aligned}$$

Collecting (2.2.18), (2.2.19) and (2.2.20), we obtain that there exist some $C_5 > 0$ and $\varepsilon_0 < \frac{1}{C_2C_4}$ such that, for all $\varepsilon < \varepsilon_0$, we have

$$\bar{\mathbb{E}}\Big[\sup_{0\le s\le T}|\tilde{X}_{s-}^{\varepsilon}|^2(1-C_2C_4\varepsilon)\Big] \le C_5,$$

which finishes the proof of the claim. □

We proceed with the main goal of this section, the proof of the large deviations principle for $(X^{\varepsilon,x})_{\varepsilon>0}$ in the superexponential regime.

Proof of Theorem 1.2.1. We verify **Condition 2.1.1.**

i) We first prove that, for every $M \ge 0$ and for every $n \in \mathbb{N}$, given $g_n, g \in S^M$ such that $\nu_T^{g_n} \to \nu_T^g$ in the vague topology of $\mathbb{M}$ as $n \to \infty$, there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}} \subset (g_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{G}^0(\nu_T^{g_{n_k}}) \to \mathcal{G}^0(\nu_T^g),$$

in the unifom topology on $C([0,T],\mathbb{R}^d)$.

We set $\tilde{X}^n := \tilde{X}^{g_n} = \mathcal{G}^0(\nu_T^{g_n})$. **Lemma 1.1.2** yields the existence of a constant $K \in (0,\infty)$ such that

$$\sup_{n\in\mathbb{N}}\sup_{0\le t\le T}|\tilde{X}_t^n| \le K. \qquad (2.2.21)$$

Since $-\nabla U$ is C^1 (**Condition 1.1.1**), $-\nabla U$ is bounded in $B_K(0)$ for some constant $C = C_K > 0$. Therefore, for all $0 \leq s \leq t \leq T$, we have

$$
\begin{aligned}
|\tilde{X}_t^n - \tilde{X}_s^n| &\leq \int_s^t |\nabla U(\tilde{X}_u^n)| du + \int_s^t \int_{\mathbb{R}^d} |z||g_n(u,z) - 1| \nu(dz) du \\
&\leq C(t-s) + \int_s^t \int_{\mathbb{R}^d} |z||g_n(u,z) - 1| \nu(dz) du.
\end{aligned}
$$

Due to (2.2.3) from **Lemma 2.2.1**, we conclude

$$
\lim_{\delta \to 0} \sup_{n \in \mathbb{N}} \sup_{|t-s| \leq \delta} |\tilde{X}_t^n - \tilde{X}_s^n| = 0.
$$

This implies that $(\tilde{X}^n)_{n \in \mathbb{N}}$ is a family of equicontinuous uniformly bounded functions in $C([0,T], \mathbb{R}^d)$. Using the *Arzelà-Ascoli theorem* (**Proposition A.1.3**) there exists a limit point in the uniform topology $\tilde{Y} \in C([0,T], \mathbb{R}^d)$ for some subsequence. Since we have the uniform estimate (2.2.21), due to the continuity of the potential $-\nabla U$ and (2.2.1) in **Lemma 2.2.1**, dominated convergence yields

$$
\tilde{Y}_t = x - \int_0^t \nabla U(\tilde{Y}_s) \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} z(g(s,z) - 1) \nu(\mathrm{d}z) \mathrm{d}s, \quad \text{for all } t \in [0,T].
$$

Since (1.1.10) has a unique continuous solution (**Lemma 1.1.2**), this shows that $\tilde{Y} = \tilde{X}^g = \mathcal{G}^0(\nu_T^g)$, which finishes the proof.

ii) We show that, given $M > 0$, $\varphi \in \mathcal{U}_+^M$ and $(\varphi_\varepsilon)_{\varepsilon > 0} \subset \mathcal{U}_+^M$ such that $\varphi_\varepsilon \Rightarrow \varphi$ in law, as $\varepsilon \to 0$, we have

$$
\mathcal{G}^0(\nu_T^g) \text{ is a limit point in law of } \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon})
$$

in $\mathbb{D}([0,T], \mathbb{R}^d)$.

In order to prove that the family $(\tilde{X}^\varepsilon)_{\varepsilon > 0}$ in $\mathbb{D}([0,T], \mathbb{R}^d)$ has a limit point with respect to the *Skorokhod topology*, as $\varepsilon \to 0$, we use the version of *Prokhorov's theorem* given in **Proposition C.1.5**. Hence, we prove that $(\tilde{X}^\varepsilon)_{\varepsilon > 0}$ is tight in $\mathbb{D}([0,T], \mathbb{R}^d)$ with respect to the Skorokhod topology using the sufficient tightness criteria of **Proposition C.2.3**.

For every $\varepsilon > 0$ and $t \in [0,T]$, let

$$
\begin{aligned}
J_t^\varepsilon &:= \int_0^t -\nabla U(\tilde{X}_s^\varepsilon) ds + \int_0^t \int_{\mathbb{R}^d} z(\varphi_\varepsilon(s,z) - 1) \nu(dz) ds \quad \text{and} \\
\bar{M}_t^\varepsilon &:= \varepsilon \int_0^t \int_{\mathbb{R}^d} z \tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds, dz),
\end{aligned}
$$

where

$$\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz) = N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz) - \frac{1}{\varepsilon}\varphi_\varepsilon \nu(dz)ds.$$

The decomposition of the measure $\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}$ is justified by the finite intensity $\nu(\mathbb{R}^d) < \infty$. We prove that, for every $\tau > 0$ there exist $\varepsilon_0 > 0$ and $\delta = \delta_\tau > 0$ such that

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{P}}\Big(\sup_{0<t-s<\delta} |J_t^\varepsilon - J_s^\varepsilon| > \tau \Big) < \tau.$$

Fix $\tau > 0$. Due to (2.2.3) of **Lemma 2.2.1**, we may choose $\delta_1 = \delta_1(\tau)$ such that, for all $0 < \delta < \delta_1$,

$$\bar{\mathbb{E}}\Big[\sup_{0<t-s<\delta} \int_s^t \int_{\mathbb{R}^d} |z||\varphi_\varepsilon(s,z) - 1|\nu(dz)ds \Big] \leq \frac{\tau^2}{4}.$$

In virtue of **Proposition 2.2.1**, let $\varepsilon_0 > 0$ and $K > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$

$$\bar{\mathbb{E}}\Big[\sup_{0\leq s\leq T} |\tilde{X}_s^\varepsilon|^2 \Big] \leq K.$$

Since $-\nabla U$ is C^1 there exists $C > 0$ such that

$$|-\nabla U(y)|^2 \leq C, \quad \text{for } y \in B_K(0).$$

This implies by *Chebyshev's inequality*, for $\delta < \delta_1$, that

$$\begin{aligned}
&\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{P}}\Big(\sup_{0<t-s<\delta} |J_t^\varepsilon - J_s^\varepsilon| > \tau \Big) \\
&= \sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{P}}\Big(\sup_{0<t-s<\delta} \Big| \int_s^t -\nabla U(\tilde{X}_r^\varepsilon)dr + \int_s^t \int_{\mathbb{R}^d} z(\varphi_\varepsilon(r,z) - 1)\nu(dz)dr \Big| > \tau \Big) \\
&\leq \sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{P}}\Big(\sup_{0<t-s<\delta} \Big| \int_s^t -\nabla U(\tilde{X}_r^\varepsilon)dr \Big| > \frac{\tau}{2} \Big) \\
&+ \sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{P}}\Big(\sup_{0<t-s<\delta} \Big| \int_s^t \int_{\mathbb{R}^d} z(\varphi_\varepsilon(r,z) - 1)\nu(dr)ds| > \frac{\tau}{2} \Big) \\
&\leq \sup_{0<\varepsilon<\varepsilon_0} \frac{4}{\tau^2}\delta^2 \bar{\mathbb{E}}\Big[\sup_{0\leq s\leq T} |\nabla U(\tilde{X}_s^\varepsilon)|^2 \Big] \\
&+ \sup_{0<\varepsilon<\varepsilon_0} \frac{2}{\tau} \bar{\mathbb{E}}\Big[\sup_{0<t-s<\delta} \int_s^t \int_{\mathbb{R}^d} |z||\varphi_\varepsilon(r,z) - 1|\nu(dz)dr \Big] \\
&\leq \frac{4\delta^2 C}{\tau^2} + \frac{\tau}{2}.
\end{aligned}$$

Choosing $\delta_2 = \delta(\tau) < \sqrt{\frac{\tau^3}{8C}}$ we have for all $0 < \delta < \delta(\tau) := \delta_1 \wedge \delta_2$,

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{P}}\Big(\sup_{0<t-s<\delta} |J_t^\varepsilon - J_s^\varepsilon| > \tau \Big) < \tau.$$

Hence, if we consider, for every $\tau > 0$, the set

$$K_\tau := \Big\{ f \in C([0,T],\mathbb{R}^d) \mid f(0)=0 \text{ and } \\ |f(t)-f(s)| < \tau 2^{-m}, \text{ for all } |t-s| \le \delta_{\tau 2^{-m}}, \text{ for every } m \in \mathbb{N} \Big\}, \quad (2.2.22)$$

it is immediate that K_τ is relatively compact in $C([0,T],\mathbb{R}^d)$ and

$$\bar{\mathbb{P}}\Big(J^\varepsilon \notin K_\tau\Big) \le \sum_{m=1}^{\infty} \tau 2^{-m} = \tau,$$

which finishes the proof that $(J^\varepsilon)_{\varepsilon>0}$ is C-tight (**Definition C.2.2**).

Concerning $(\bar{M}^\varepsilon)_{\varepsilon>0}$, we conclude by (2.2.2) of **Lemma 2.2.1**

$$\begin{aligned} \bar{\mathbb{E}}\Big[[\bar{M}^\varepsilon]_T\Big] &= \varepsilon \bar{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{R}^d} |z|^2 \varphi_\varepsilon(s,z)\nu(dz)ds\Big] \\ &\le \varepsilon \sup_{g \in S^M} \bar{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{R}^d} |z|^2 g(s,z)\nu(dz)ds\Big] \to 0 \quad \text{as } \varepsilon \to \infty, \end{aligned} \quad (2.2.23)$$

which implies that $([\bar{M}^\varepsilon])_{\varepsilon>0}$ is C-tight (**Definition C.2.2**.) Due to **Proposition C.2.3**, the laws of the family $\tilde{X}^\varepsilon_t = x + J^\varepsilon_t + M^\varepsilon_t$ are tight in $\mathbb{D}([0,T],\mathbb{R}^d)$.
Using *Prokhorov's Theorem* (**Proposition C.1.5**), there exists a weak limit of $(\tilde{X}^\varepsilon, J^\varepsilon, M^\varepsilon)$. The version of *Skorokhod's theorem* stated in **Proposition C.1.7** implies that there exists $(\tilde{X}, \tilde{\varphi}, 0)$ defined on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ such that $(\tilde{X}^\varepsilon, J^\varepsilon, M^\varepsilon)$ converges to $(\tilde{X}, \tilde{\varphi}, 0)$ $\bar{\mathbb{P}}$-a.s. For every $\varepsilon > 0$ and $t \in [0,T]$,

$$\tilde{X}^\varepsilon_t = x - \int_0^t \nabla U(\tilde{X}^\varepsilon_s)ds + \int_0^t \int_{\mathbb{R}^d} z(\varphi_\varepsilon(s,z)-1)\nu(dz)ds + M^\varepsilon_t. \quad (2.2.24)$$

Due to the continuity condition of the potential U and (2.2.1), using dominated convergence theorem we can pass to the pointwise limit $\tilde{X}^\varepsilon_t \to \tilde{X}_t$ $\bar{\mathbb{P}}$-a.s. in (2.2.24). Hence, we conclude that $(\tilde{X}_s)_{0\le t\le T}$ satisfies, for all $t \in [0,T]$,

$$\tilde{X}_t = x - \int_0^t \nabla U(\tilde{X}_s)ds + \int_0^t \int_{\mathbb{R}^d} z(\tilde{\varphi}(s,z)-1)\nu(dz)ds.$$

Therefore, due to **Lemma 1.1.2** and since the equation above admits a unique solution in $C([0,T],\mathbb{R}^d)$, we conclude that $\tilde{X} = \mathcal{G}^0(\nu_T^{\tilde{\varphi}})$. Since φ and $\tilde{\varphi}$ are indistinguishable in law and that almost sure convergence implies convergence in probability, which implies therefore convergence in law, we finished proving that

$$\mathcal{G}^0(\nu_T^\varphi) \text{ is a weak limit point of } \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}).$$

$\square$

2.3 The asymptotic first exit time

2.3.1 Continuity properties of the cost functional

Proposition 2.3.1. *There exist $M > 0$, $\rho > 0$ and $T : [0, \rho] \longrightarrow \mathbb{R}^+$ such that $\lim_{\rho \to 0} T(\rho) = 0$ satisfying the following.*
For all $x_0, y_0 \in \mathbb{R}^d$ such that $|x_0 - y_0| \le \rho$ there exist $\Phi \in C([0, T(\rho)], \mathbb{R}^d)$ and $g \in S^M$ such that $\Phi(T(\rho)) = y_0$ and solving

$$\Phi(s) = x_0 - \int_0^s \nabla U(\Phi(r))\mathrm{d}r + \int_0^s \int_{\mathbb{R}^d} z(g(r,z) - 1)\nu(\mathrm{d}z)\mathrm{d}r, \quad 0 \le s \le T(\rho). \tag{2.3.1}$$

In particular, for $\bar{V}$ defined in (1.2.4) we have $\bar{V} < \infty$.

Proof. We construct functions $\Phi \in C([0, T], \mathbb{R}^d)$ and $g \in \mathbb{S}$ such that (2.3.1) holds. By symmetry of the measure ν, for every vector $x \in \mathbb{R}^d$ there exists a measurable function $f^x : \mathbb{R}^d \longrightarrow [0, \infty)$ such that

$$x = \int_{\mathbb{R}^d} z f^x(z)\nu(dz).$$

For instance, fixed $\delta > 0$, we choose the function

$$f^x(z) = \frac{e^{|z|^\alpha}}{\lambda^d(B_\delta(x))} \mathbf{1}_{B_\delta(x)}(z).$$

where λ^d is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ Fixed the points $x, y \in \mathbb{R}^d$, we construct a linking path between them in the following way.
Let u^x, u^y solutions of the deterministic differential equation

$$\dot{u}(t) = -\nabla U(u(t))$$

with initial conditions x and y respectively.
Given $\delta > 0$, since we assume **Condition 1.1.1**, let $s_1 > 0$ such that $z_1 = u^x(s_1)$ and for all $t \ge s_1$ $u^x(t) \in B_\delta(0)$. Let $s_2 > 0$ be such that $z_2 = u^y(s_2)$ and for all $t \ge s_2$, $u^y(t) \in B_\delta(0)$.
Let us assume that $z_1 \ne z_2$. Fix $\rho > 0$.
Construct

$$\Phi : [0, 2\rho s_2] \longrightarrow \mathbb{R}^d$$

$$\Phi(s) = \begin{cases} u^x(\frac{s}{\rho}), & s \in [0, \rho s_1], \\ z_1 + \dfrac{\frac{s}{\rho} - s_1}{s_2 - s_1}(z_2 - z_1), & s \in [\rho s_1, \rho s_2], \\ z_2 + \displaystyle\int_{s_2}^{\frac{s}{\rho}} \nabla U(u^y(r - s_2))dr, & s \in [\rho s_2, 2\rho s_2]. \end{cases} \tag{2.3.2}$$

For this path that links x and y, we construct a control $g \in S^M$, for some $M > 0$, satisfying (2.3.1), in the following way.

i) For $s \in [0, \rho s_1]$ we choose $g := 1$.

ii) For $s \in [\rho s_1, \rho s_2]$, we write

$$Q_{z_1,z_2}(s) = z_1 + \frac{1}{\rho(s_2 - s_1)}(z_2 - z_1) + \nabla U\Big(z_1 + \frac{\frac{s}{\rho} - s_1}{s_2 - s_1}(z_2 - z_1)\Big)$$

and for $(s, z) \in [\rho s_1, \rho s_2] \times \mathbb{R}^d$,

$$g(s, z) := 1 + \frac{e^{|z|^\alpha}}{\lambda^d(B_1(Q_{z_1,z_2}(s)))} \mathbf{1}_{B_1(Q_{z_1,z_2}(s))}(z).$$

The control g defined for $s \in [\rho s_1, \rho s_2]$ is bounded by a certain constant C, due to the fact of $z_1, z_2 \in B_\delta(0)$ and the continuity of ∇U.

iii) For $s \in [\rho s_2, 2\rho s_2]$, we write

$$P_{z_2,y}(s) = \frac{1}{\rho}\nabla U\Big(u^y(s - s_2)\Big).$$

We define, for every $(s, z) \in [\rho s_2, 2\rho s_2] \times \mathbb{R}^d$,

$$g(s, z) := 1 + \frac{e^{|z|^\alpha}}{\lambda^d(B_1(P_{z_2,y}(s)))} \mathbf{1}_{B_1(P_{z_2,y}(s))}(z).$$

$P_{z_2,y}(s)$, $s \in [\rho s_2, 2\rho s_2]$, is bounded by construction, due to the continuity of u^y in the compact $[\rho s_2, 2\rho s_2]$ and due to the continuity of ∇U. Therefore, the control function g is bounded.

By construction, the control function $g : [0, 2\rho s_2] \times \mathbb{R}^d \longrightarrow [0, \infty)$ is bounded, which implies that there exists $C > 0$ such that $\ell(g(s, z)) \leq C$ for every $(s, z) \in [0, 2\rho s_2] \times \mathbb{R}^d$. Hence

$$\bar{V} \leq \int_0^{2\rho s_2} \int_{\mathbb{R}^d} \ell(g(s, z))\nu(dz)ds \leq 2C\rho s_2,$$

where $\ell(g(s, z)) = g(s, z) \ln g(s, z) - g(s, z) + 1$. Furthermore, $\Phi \in C([0, T], \mathbb{R}^d)$ links x and y. Φ and $g \in \mathbb{S}$, defined as above, solve the equation,

$$\Phi(s) = x + \int_0^s -\nabla U(\Phi_r)\mathrm{d}r + \int_0^s \int_{\mathbb{R}^d} z(g(r, z) - 1)\nu(\mathrm{d}z)\mathrm{d}r \quad 0 \leq s \leq 2\rho s_2.$$

Choosing $T(\rho) = 2\rho s_2$ the second statement follows.

$\square$

Corollary 2.3.1. *For any $\delta > 0$, there exists $\rho > 0$ such that:*

- *(1)* $\sup_{|x|,|y|\le\rho} \inf_{t\in[0,1]} V(x,y,t) < \delta$

- *(2)* $\sup_{\{x,y:\inf_{z\in D^c}|x-z|+|y-z|\le\rho\}} \inf_{t\in[0,1]} V(x,y,t) < \delta$

Proof. 1. Let us fix $\delta > 0$. For fixed $\rho > 0$ and given $x, y \in \mathbb{R}^d$ such that $|x|, |y| \le \rho$ we consider the straight line that links x and y,

$$\Phi(t) = x + t\frac{y-x}{\rho}, \quad t \in [0,\rho].$$

For every $t \in [0,1]$ we write

$$P_{x,y}(t) = \frac{(y-x)}{\rho} + \nabla U(\Phi(t)).$$

For every $(s,z) \in [0,1] \times \mathbb{R}^d$ we define the function

$$g : [0,\rho] \times \mathbb{R}^d \longrightarrow [0,\infty),$$

$$g(s,z) = 1 + \frac{e^{|z|^\alpha}}{\lambda^d(B_1(P_{x,y}(s)))}\mathbf{1}_{B_1(P_{x,y}(s))}(z),$$

where λ^d is the d-dimensional Lebesgue measure.
By construction of Φ and g we have, for every $t \in [0,\rho]$,

$$\Phi(t) = x - \int_0^t \nabla U(\Phi(s))ds + \int_0^t \int_{\mathbb{R}^d} z(g(s,z)-1)\nu(dz)ds.$$

Due to the continuity of ∇ and Φ, the function $P_{x,y}$ is bounded in $[0,\rho]$. Therefore the function g is bounded in $[0,\rho] \times \mathbb{R}^d$ which implies that $\ell(g)$ is bounded by a certain constant $C > 0$. Hence, $g \in S^M$ for certain $M > 0$.
Choosing $T(\rho) = \rho$ we have

$$V(x,y,T(\rho)) \le C\nu(\mathbb{R}^d)\rho.$$

Choosing $\rho = \frac{\delta}{C\nu(\mathbb{R}^d)}$ the first statement follows.

2. The conclusion of the second statement is immediate.

$\square$

2.3.2 Asymptotic upper bound for the exit of a ball and implications

Let us fix $x \in \mathbb{R}^d$. For every $\varepsilon > 0$, due to **Theorem 1.1.1**, let $(X^{\varepsilon,x})_{t\geq 0}$ the unique solution of (1.1.4) in the sense of **Definition 1.1.1**. We write, for every $\varepsilon > 0$,

$$X_t^{\varepsilon,x} = x - \int_0^t \nabla U(X_s^{\varepsilon,x})ds + \varepsilon \tilde{L}_t^{\varepsilon}, \quad t \geq 0,$$

with

$$\tilde{L}_t^{\varepsilon} := \int_0^t \int_{\mathbb{R}^d} z\tilde{N}^{\frac{1}{\varepsilon}}(ds, dz), \quad t \geq 0.$$

For every $\varepsilon > 0$, the stochastic process $(\tilde{L}_t^{\varepsilon})_{t\geq 0}$ is a compensated compound Poisson process with jump intensity $\varepsilon^{-1}\nu(\mathbb{R}^d)$. We proceed now to the characterization of the jumps $(W_i^{\varepsilon})_{i\in\mathbb{N}}$ and of the jumping times $(T_i^{\varepsilon})_{i\in\mathbb{N}}$ of $(\tilde{L}_t^{\varepsilon})_{t\geq 0}$

Recursively, for every $\varepsilon > 0$ and $n \in \mathbb{N}_1$, we define the jump times

$$T_n^{\varepsilon} := \sum_{i=1}^{n} \tau_i^{\varepsilon}$$

in the following way:

$$\begin{aligned} T_1^{\varepsilon} = \tau_1^{\varepsilon} &:= \inf\{t > 0 \mid \Delta_t X^{\varepsilon} \neq 0\}, \\ \tau_{i+1}^{\varepsilon} &:= \inf\{t > 0 \mid \Delta_{t+T_i^{\varepsilon}} X^{\varepsilon} \neq 0\}, \\ \tau_i^{\varepsilon} &:= T_i^{\varepsilon} - T_{i-1}^{\varepsilon}, \quad i \in \{1, \ldots, n\} \end{aligned} \tag{2.3.3}$$

Remark 2.3.1 (**About the jumps and jumping times times of** $(\tilde{L}_t^{\varepsilon})_{t\geq 0}$).

i) For every $\varepsilon > 0$, the jumps of the process $(\varepsilon\tilde{L}_t^{\varepsilon})_{t\geq 0}$ are of the form $(W_i^{\varepsilon})_{i\in\mathbb{N}_1} \equiv (\varepsilon W_i)_{i\in\mathbb{N}_1}$, where $(W_i)_{i\in\mathbb{N}_1}$ is a sequence of random variables i.i.d. defined on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ with law $\frac{\nu}{\beta}$, with $\beta := \nu(\mathbb{R}^d) < \infty$ (see **Remark 1.1.5**).

ii) The waiting times $(\tau_i^{\varepsilon})_{i\in\mathbb{N}}$ are of the form $(\varepsilon\tau_i)_{i\in\mathbb{N}_1}$, where τ_i is a random variable defined on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ with exponential law, $\tau_i \sim EXP(\beta)$, for all $i \in \mathbb{N}_1$.

iii) For all $x \in \mathbb{R}^d$ and $\varepsilon > 0$, since $(\tilde{L}_t^{\varepsilon})_{t\geq 0}$ is a compensated compound Poisson process with finite intensity $\varepsilon^{-1}\nu(\mathbb{R}^d)$ we have that between the jumping times the process $(X^{\varepsilon,x})_{t\in[0,T]}$ follows a deterministic motion . Furthermore, we have only only the occurrence of finitely many jumps in any finite time interval.

Theorem 2.3.1. *We assume that the jump measure is of the form*

$$\nu(dz) = e^{-|z|^{\alpha}}dz \text{ for some } \alpha > 0.$$

For every $R > 0$, there exists $C(R) > 0$ such that

$$\limsup_{\varepsilon \to 0} \varepsilon^{\alpha} \ln \bar{\mathbb{P}}\Big(\sup_{t \geq 0} |X_t^{\varepsilon,x}| \geq R \Big) \leq -C(R) \quad \text{for all } |x| \leq \frac{R}{2} \tag{2.3.4}$$

and $\lim_{R \to +\infty} C(R) = +\infty$.

Remark 2.3.2. *For every $|x| < \frac{R}{2}$, the theorem above shows that the asymptotics of the exit of $(X^{\varepsilon,x})_{\varepsilon>0}$ from a ball of radius R centered in the stable state of the underlying dynamical system $X^{0,x}$ follows two different regimes, according to the distinction of the lightness parameter α of the jump measure ν.*

1. *If $\alpha \geq 1$ and ν* ***is a superexponential light jump measure****, the asymptotics of the exit from the ball of radius $R > 0$ centered in 0 follows a large deviations scale, since*

$$\begin{aligned} \varepsilon \ln \bar{\mathbb{P}}\Big(\sup_{t \geq 0} |X_t^{\varepsilon,x}| \geq R \Big) &\leq \frac{-C(R)}{\varepsilon^{\alpha-1}} \\ &\leq -C(R). \end{aligned}$$

2. *If $\alpha \in (0,1)$ and ν* ***is a subexponential light jump measure****, the asymptotics of the exit from the ball of radius $R > 0$ centered in 0 follows a moderate deviations scale.*

Proof. Fix $R > 0$ and let $|x| \leq \frac{R}{2}$. We have

$$\begin{aligned} \bar{\mathbb{P}}\Big(\sup_{t \geq 0} |X_t^{\varepsilon,x}| \geq R \Big) &\leq \bar{\mathbb{P}}\Big(\sup_{t \geq 0} |X_t^{\varepsilon,x} - w(t;x)| \geq \frac{R}{2} \Big) \\ &\quad + \bar{\mathbb{P}}\Big(\sup_{t \geq 0} |w(t;x)| \geq \frac{R}{2} \Big), \end{aligned}$$

where $w(t;x)$ is the unique solution of

$$\begin{cases} \dot{w}(t) = -\nabla U(w(t)) - \displaystyle\int_{\mathbb{R}^d} z\nu(dz) \\ w(0) = x. \end{cases} \tag{2.3.5}$$

We observe on the event $\{t \in (0, \varepsilon T_1]\}$ we have

$$X_t^{\varepsilon,x} = w(t;x) + \varepsilon W_1 \mathbf{1}_{\{t = \varepsilon T_1\}}.$$

More generally, for $\{t \in (\varepsilon T_n, \varepsilon T_{n+1}]\}$, $n \in \mathbb{N}_1$,

$$X_t^{\varepsilon,x} = w\Big(t - \varepsilon T_n, X_{\varepsilon T_n}^{\varepsilon,x}\Big) + \varepsilon W_{n+1} \mathbf{1}_{\{t = \varepsilon T_{n+1}\}}.$$

Given $x, y \in \mathbb{R}^d$ and $w(.;x)$ and $w(.;y)$ solutions of (2.3.5) with initial conditions x and y respectively, due to (1.1.2), we have, for all $a > 1$ and $t \geq 0$,

$$
\begin{aligned}
\sqrt{a}|w(t;x) - w(t;y)|^2 &= a|x-y|^2 \\
&\quad + a\int_0^t 2\langle(-\nabla U)(w(s;x)) - (-\nabla U)(w(s;y)), w(s;x) - w(s;y)\rangle ds \\
&\leq a|x-y|^2 - 2a\eta\int_0^t |w(s;x) - w(x;y)|^2 ds. \qquad (2.3.6)
\end{aligned}
$$

Using Gronwall's inequality, we derive, for all $a > 1$ and $t \geq 0$,

$$
|w(t;x) - w(t;y)|^2 \leq |x-y|^2 \exp(-2a\eta t)\sqrt{a},
$$

and we observe that $\exp(-2a\eta t)\sqrt{a} \to 0 \quad \text{as } a \to \infty$. For convenience we write $X_t^{\varepsilon,x} = X_t^{\varepsilon}(x)$, for every $t \geq 0$ and $x \in \mathbb{R}^d$.
For every $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}_1$, we conclude

$$
\begin{aligned}
\left|X^{\varepsilon}_{\varepsilon T_{n+1}}(x) - w(\varepsilon T_{n+1};y)\right|^2 &\leq 2|w(\varepsilon\tau_{n+1}; X^{\varepsilon}_{\varepsilon T_n}(x)) - w(\varepsilon\tau_{n+1}; w(\varepsilon T_n;y)|^2 + 2\varepsilon^2|W_{n+1}|^2 \\
&\leq q|X^{\varepsilon}_{\varepsilon T_n}(x) - w(\varepsilon T_n;y)|^2 + 2\varepsilon^2|W_{n+1}|^2
\end{aligned}
$$

where, for every $\varepsilon > 0$,

$$
q = q_n = 2\exp(-2a_n\eta\varepsilon\tau_{n+1}) = 2\exp(-2) < 1,
$$

choosing $a_n = \frac{1}{\eta\varepsilon\tau_{n+1}}$. We fix $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ we have $a_n > 1$.
For every $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we set

$$
b_{n+1} = \left|X^{\varepsilon}_{\varepsilon T_{n+1}}(x) - w(\varepsilon T_{n+1};y)\right|^2
$$

and derive the following recurrence relation,

$$
\begin{cases}
b_{n+1} \leq q b_n + 2\varepsilon^2|W_{n+1}|^2 & \text{for all } n \geq 0, \\
b_0 = |x-y|^2.
\end{cases}
$$

Hence, by induction in $n \in \mathbb{N}$, we have

$$
b_n \leq |x-y|^2 + 2\varepsilon^2 \sum_{i=1}^n q^{n-i}|W_i|^2.
$$

In case of $x = y$ it follows, for all $n \in \mathbb{N}$,

$$
|X^{\varepsilon,x}_{\varepsilon T_n} - w(\varepsilon T_n;x)|^2 \leq 2\varepsilon^2 \sum_{i=1}^n q^{n-i}|W_i|^2.
$$

We note that

$$\sum_{i=1}^{n} q^{n-i}|W_i|^2 \stackrel{d}{=} \sum_{i=1}^{n} q^i |W_1|^2,$$

since $(W_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with law $\frac{\nu}{\beta}$, where $\beta = \nu(\mathbb{R}^d) < \infty$. We show the $\bar{\mathbb{P}}$- a.s. convergence of the random variable

$$S_1^2 := \lim_{n\to\infty} \sum_{i=1}^{n} q^i |W_1|^2.$$

In order to apply *Kolmogorov's three series theorem* (**Proposition C.1.9**), we show the convergence of

$$\sum_{n=1}^{\infty} \bar{\mathbb{E}}\Big[q^n|W_1|^2\Big] \text{ and } \sum_{n=1}^{\infty} \mathrm{var}\Big[q^n|W_1|^2\Big].$$

It is clear that

$$\begin{aligned}\bar{\mathbb{E}}\Big[q^n|W_1|^2\Big] &= q^n \bar{\mathbb{E}}\Big[|W_1|^2\Big] \\ &= 2q^n \int_0^{\infty} r\bar{\mathbb{P}}(|W_1| > r)dr,\end{aligned}$$

and, for certain $c_d > 0$,

$$\begin{aligned}\bar{\mathbb{P}}(|W_1| > r) &= \frac{1}{\beta}\int_{B_r^c(0)} e^{-|z|^{\alpha}} dz \\ &= \frac{c_d}{\beta}\int_r^{\infty} e^{-x^{\alpha}} x^{d-1} dx, \quad x = |z| \\ &= \frac{c_d}{\alpha\beta}\Gamma\Big(\frac{d}{\alpha}, r^{\alpha}\Big),\end{aligned}$$

where $\Gamma(s,y)$ is the incomplete Euler's Γ-function,

$$\Gamma(s,y) = \int_y^{\infty} x^{s-1}e^{-x}dx, \quad s,y \in \mathbb{R}.$$

Due to the asymptotic property of the Γ function (see chapter 6 in *Abramovitz et al.* (1964)),

$$\frac{\Gamma(s,y)}{y^{s-1}\exp(-y)} \to 1, \qquad y \to \infty, \tag{2.3.7}$$

there exists $r_1 > 0$ and $C_1 > 0$ such that for $r > r_1$ we have

$$\bar{\mathbb{P}}(|W_1| > r) \le \frac{C_1 c_d}{\alpha\beta} r^{d-\alpha} e^{-r^{\alpha}}.$$

This yields, after change of variables,

$$\begin{aligned}\sum_{n=1}^{\infty}\bar{\mathbb{E}}[q^n|W_1|^2] &\leq \sum_{n=1}^{\infty}\frac{2q^nC_1c_d}{\alpha\beta}\int_0^{\infty} z^{2+d-\alpha-1}e^{-z^\alpha}dz\\ &= \sum_{n=1}^{\infty}\frac{2q^nC_1c_d}{\alpha^2\beta}\Gamma\Big(\frac{d+2-\alpha}{\alpha}\Big)\\ &= \frac{2C_1c_d\Gamma q\Big(\frac{d+2-\alpha}{\alpha}\Big)}{\alpha^2\beta(1-q)}\\ &< \infty.\end{aligned}$$

Since

$$\mathrm{var}[q^n|W_1|^2] = q^{2n}\Big(\bar{\mathbb{E}}|W_1|^4 - (\bar{\mathbb{E}}|W_1|^2)^2\Big),$$

the second term in the right hand side of the following sum is finite due to the calculations above. Therefore,

$$\sum_{n=1}^{\infty}\ \mathrm{var}\ [q^n|W_1|^2] = \sum_{n=1}^{\infty}q^{2n}\Big(\bar{\mathbb{E}}|W_1|^4 - (\bar{\mathbb{E}}|W_1|^2)^2\Big).$$

Similarly to what was made before, there exist $C_2, c_d > 0$ such that, after change of variables,

$$\begin{aligned}\sum_{n=1}^{\infty}q^{2n}\bar{\mathbb{E}}\Big[|W_1|^4\Big] &= \sum_{n=1}^{\infty}4q^{2n}\int_0^{\infty} r^3\bar{\mathbb{P}}(|W_1|>r)dr\\ &\leq \sum_{n=1}^{\infty}\frac{4q^{2n}c_d}{\alpha^2\beta}\Gamma\Big(\frac{d+4-\alpha}{\alpha}\Big)\\ &\leq \frac{4c_dC_2q\Gamma\Big(\frac{d+4-\alpha}{\alpha}\Big)}{\alpha^2\beta(1-q^2)}\\ &< \infty.\end{aligned}$$

By *Kolmogorov's three series theorem* (**Proposition C.1.9**), we have almost surely the monotonic convergence in $\bar{\mathbb{P}}$

$$S_1^2 := \lim_{n\to\infty}\sum_{i=1}^{n}q^{n-i}|W_i|^2 \overset{d}{=} \sum_{i=1}^{\infty}q^i|W_1|^2.$$

Hence, due to (2.3.6), we derive, for all $x \in \mathbb{R}^d$

$$\sup_{t\geq 0}|X_t^{\varepsilon,x} - w(t;x)|^2 = \sup_{n\in\mathbb{N}}|X_{\varepsilon T_n}^{\varepsilon,x} - w(\varepsilon T_n;x)|^2 \leq 2\varepsilon^2S_1^2. \tag{2.3.8}$$

Therefore, for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\bar{\mathbb{P}}\Big(\sup_{n\geq 0}|X^{\varepsilon,x}_{\varepsilon T_n} - w(\varepsilon T_n;x)|^2 \geq \frac{R^2}{4}\Big)\\
&\leq \bar{\mathbb{P}}\Big(2\varepsilon^2 S_1^2 \geq \frac{R^2}{4}\Big)\\
&= \bar{\mathbb{P}}\Big(\sum_{i=1}^{\infty} q^i |W_i|^2 \geq \frac{R^2}{8\varepsilon^2}\Big)\\
&\leq \sum_{i=1}^{\infty} \bar{\mathbb{P}}\Big(q^i|W_i|^2 \geq \frac{(1-\sqrt{q})(\sqrt{q})^i}{\sqrt{q}}\frac{R^2}{8\varepsilon^2}\Big)\\
&= \sum_{i=1}^{\infty} \bar{\mathbb{P}}\Big(|W_1| \geq \frac{R}{2\sqrt{2}}\sqrt{\frac{1-\sqrt{q}}{\sqrt{q}(\sqrt{q})^i}}\varepsilon^{-1}\Big)\\
&= \sum_{i=1}^{\infty} \bar{\mathbb{P}}(|W_1| \geq C(R,q,i)\varepsilon^{-1})\\
&= \sum_{i=1}^{\infty} \frac{c_d}{\beta\alpha}\Gamma\Big(\frac{d}{\alpha}, (C(R,q,i)\varepsilon^{-1})^{\alpha}\Big),
\end{aligned}$$

where

$$C(R,q,i) := \frac{R}{2\sqrt{2}}\sqrt{\frac{1-\sqrt{q}}{\sqrt{q}(\sqrt{q})^i}}, \quad \text{for all } i \in \mathbb{N}. \tag{2.3.9}$$

The asymptotic behaviour of the incomplete Γ-function (see chapter 6 in *Abramovitz et al.* (1964))

$$\frac{\Gamma(s,y)}{y^{s-1}\exp(-y)} \to 1, \qquad y \to \infty$$

yields some $C_3 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned}
&\bar{\mathbb{P}}(\sup_{n\geq 0}|X^{\varepsilon,x}_{\varepsilon T_n} - w(\varepsilon T_n;x)|^2 \geq \frac{R^2}{4})\\
&\leq \frac{C_3}{\beta\alpha}\sum_{i=1}^{\infty}(C(R,q,i)\varepsilon^{-1})^{d-\alpha}e^{-(C(R,q,i)\varepsilon^{-1})^{\alpha}}.
\end{aligned}$$

The expression above converges. The term which converges slowest in the right hand side of the previous expression as a function in ε is the first one, $i = 1$. Hence this term dominates the asymptotics, which implies, for some $C_4 > 0$ and $\tilde{C}(R) = C(R,q,1) > 0$,

$$\begin{aligned}
&\bar{\mathbb{P}}\Big(\sup_{n\in\mathbb{N}}|X^{\varepsilon,x}_{\varepsilon T_n} - w(\varepsilon T_n;x)|^2 \geq \frac{R^2}{4}\Big)\\
&\leq \frac{C_4}{\beta\alpha}(C(R)\varepsilon^{-1})^{d-\alpha}e^{-(C(R)\varepsilon^{-1})^{\alpha}}.
\end{aligned} \tag{2.3.10}$$

In the next estimate, the second term tends to 0 and, since $\alpha \geq 1$, we have, for some $\varepsilon_0 > 0$ and for every $\varepsilon < \varepsilon_0$,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^\alpha \ln \bar{\mathbb{P}}(\sup_{t\geq 0} |X_t^{\varepsilon,x}| \geq R) \\
&\leq \lim_{\varepsilon\to 0} \varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{t\geq 0} |X_t^{\varepsilon,x} - w(t;x)| \geq \frac{R}{2}\Big) + \lim_{\varepsilon\to 0} \varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{t\geq 0} |w(t;x)| \geq \frac{R}{2}\Big) \\
&\leq -C(R),
\end{aligned}
\tag{2.3.11}
$$

where $C(R) = (\tilde{C}(R))^\alpha \to \infty$ as $R \to \infty$.

In what follows, we justify that

$$
\mathbb{P}(\sup_{t\geq 0} |w(t;x)|^2 \geq \frac{R^2}{4}) = 0.
$$

For every $x \in \mathbb{R}^d$ such that $|x| \leq \frac{R}{2}$, $t \geq 0$, due to (1.1.2), we have, due to the symmetry of ν,

$$
\begin{aligned}
\frac{d}{dt}|w(t;x)|^2 &= 2\langle w(t;x), \dot{w}(t;x)\rangle \\
&= 2\langle -\nabla U(w(t;x)), w(t;x)\rangle - 2\int_{\mathbb{R}^d} \langle z, w(t;x)\rangle \nu(dz) \\
&= 2\langle -\nabla U(w(t;x)), w(t;x)\rangle - 2\langle \int_{\mathbb{R}^d} z\nu(dz), w(t;x)\rangle \\
&= 2\langle -\nabla U(w(t;x)), w(t;x)\rangle \\
&\leq -2\eta |w(t;x)|^2.
\end{aligned}
$$

Using *Gronwall's inequality*, we conclude, for $|x| \leq \frac{R}{2}$,

$$
|w(t;x)|^2 \leq |x|^2 e^{-2\eta t} < \frac{R^2}{4}, \qquad \text{for all } t \geq 0.
$$

This finishes the proof. $\square$

We prove that the large deviations principle for $(X^{\varepsilon,x})_{\varepsilon>0}$ is uniform with respect to the initial condition $x \in \mathbb{R}^d$. For every $T > 0$, we always consider the càdlàg space $\mathbb{D}([0,T], \mathbb{R}^d)$ endowed with the Skorokhod topology (1.2.3). We refer the reader to **section C.2** for a brief survey on the càdlàg space and the Skorokhod topology.

Proposition 2.3.2 (Uniform Large Deviations Principle.). *Given $T > 0$ and $x \in \mathbb{R}^d$, let $F \subset \mathbb{D}([0,T], \mathbb{R}^d)$ be closed and $G \subset \mathbb{D}([0,T], \mathbb{R}^d)$ open with respect to the Skorokhod* *topology. Then we have*

$$a)\ \limsup_{\varepsilon\to 0, y\to x} \varepsilon \ln \bar{\mathbb{P}}(X^{\varepsilon,y} \in F) \leq -\inf_{f\in F} \mathbb{J}(f)_{x,T},$$
$$b)\ \liminf_{\varepsilon\to 0 y\to x} \varepsilon \ln \bar{\mathbb{P}}(X^{\varepsilon,y} \in G) \geq -\inf_{g\in G} \mathbb{J}(g)_{x,T}.$$

Proof. In view of **Theorem D.1.1**, it is enough to show that $(X^{\varepsilon,x})_{\varepsilon>0}$ and $(X^{\varepsilon,x_\varepsilon})_{\varepsilon>0}$ are exponentially equivalent, with $(x_\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}^d$ converging to x as $\varepsilon \to 0$.

Fix $\delta > 0$. We show that

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \bar{\mathbb{P}}\Big(d_{J_1}(X^{\varepsilon,x_\varepsilon}, X^{\varepsilon,x}) > \delta\Big) = -\infty,$$

where d_{J_1} is defined in (1.2.3). Since the J_1 topology is finner than the uniform topology on $\mathbb{D}([0,T],\mathbb{R}^d)$,

$$\bar{\mathbb{P}}\Big(d_{J_1}(X^{\varepsilon,x_\varepsilon}, X^{\varepsilon,x}) > \delta\Big) \leq \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |X_t^{\varepsilon,x_\varepsilon} - X_t^{\varepsilon,x}| > \delta\Big).$$

Let $w(.;x)$ the unique continuous solution of the ordinary differential equation

$$\begin{cases} \dot{w}(t) = -\nabla U(w(t)) - \displaystyle\int_{\mathbb{R}^d} z\nu(dz) \\ w(0) = x, \end{cases}$$

and, for every $\varepsilon > 0$ $w(x_\varepsilon;.)$, the unique continuous solution of the same ordinary differential equation but with initial condition x_ε. Due to (2.3.6) and **Condition 1.1.1** we have, for all $\varepsilon > 0$,

$$\sup_{t\in[0,T]} |w(t;x) - w(t;x_\varepsilon)|^2 \leq |x - x_\varepsilon|^2. \tag{2.3.12}$$

Setting $R = \frac{\delta}{3}$ in **Theorem 2.3.1**, (2.3.8) and (2.3.10) imply that there exist $C(R) > 0$ and $\varepsilon_0 > 0$ small enough such that, for every $0 < \varepsilon < \varepsilon_0$, we conclude

$$\varepsilon \ln \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |X^{\varepsilon,x} - w(t;x)| \geq \frac{\delta^2}{3}\Big) \leq -\frac{C(R)}{\varepsilon^{\alpha-1}} \text{ and}$$
$$\varepsilon \ln \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |X^{\varepsilon,x_\varepsilon} - w(t;x_\varepsilon)| \geq \frac{\delta}{3}\Big) \leq -\frac{C(R)}{\varepsilon^{\alpha-1}}. \tag{2.3.13}$$

Hence, choosing $\varepsilon_0 > 0$ small enough such that $|x_\varepsilon - x|^2 < \frac{\delta^2}{3}$ for $0 < \varepsilon < \varepsilon_0$, from (2.3.8) and (2.3.12) we have

$$\begin{aligned}
\varepsilon \ln \bar{\mathbb{P}}\Big(d_{J_1}(X^{\varepsilon,x_\varepsilon}, X^{\varepsilon,x}) > \delta\Big) &\leq \varepsilon \ln \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |X_t^{\varepsilon,x_\varepsilon} - X_t^{\varepsilon,x}| > \delta\Big) \\
&\leq \varepsilon \ln \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |X_t^{\varepsilon,x_\varepsilon} - w(t;x_\varepsilon)| > \frac{\delta}{3}\Big) \\
&+ \varepsilon \ln \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |X_t^{\varepsilon,x} - w(t;x)| > \frac{\delta}{3}\Big) \\
&+ \varepsilon \ln \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |w(t;x) - w(t;x_\varepsilon)|^2 > \frac{\delta}{3}\Big) \\
&+ 2\varepsilon \ln 2,
\end{aligned}$$

which tends to $-\infty$ as $\varepsilon \to 0$.
This finishes the proof. □

Corollary 2.3.2 (Uniform LDP in compact sets of initial states). *Let $T > 0$, $K \subset \mathbb{R}^d$ be compact, $F \subset \mathbb{D}([0,T], \mathbb{R}^d)$ closed, $G \subset \mathbb{D}([0,T], \mathbb{R}^d)$ open with respect to the J_1 topology and $x \in \mathbb{R}^d$. Then it follows*

$$\begin{aligned}
&a) \limsup_{\varepsilon\to 0} \sup_{y\in K} \varepsilon \ln \bar{\mathbb{P}}(X^{\varepsilon,y} \in F) \leq - \inf_{y\in K, f\in F} \mathbb{J}(f)_{y,T} \text{ and} \\
&b) \liminf_{\varepsilon\to 0} \inf_{y\in K} \varepsilon \ln \bar{\mathbb{P}}(X^{\varepsilon,y} \in G) \geq - \inf_{y\in K, g\in G} \mathbb{J}(g)_{y,T}.
\end{aligned}$$

Proof. We prove the upper bound.
Fix K and F according to the statement. We set $J_K(F) = \inf_{y\in K, f\in F} \mathbb{J}_{y,T}(f)$. For fixed $\delta > 0$ we define $J_K^\delta(F) := \min\{J_K(F) - \delta, \frac{1}{\delta}\}$. For any $x \in K$, **Proposition 2.3.2** yields the existence of $\varepsilon_x > 0$ such that, for $\varepsilon < \varepsilon_x$,

$$\varepsilon \ln \sup_{y\in B_{\varepsilon_x}(x)} \bar{\mathbb{P}}(X^{\varepsilon,y} \in F) \leq -J_K^\delta(F).$$

The compactness of K permits the choice of a finite open subcover $(B_{\varepsilon_{x_i}}(x_i))_{i=1,\dots,n}$ of K, for some $x_1, \dots, x_n \in K$ and $n \in \mathbb{N}$. This implies

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \sup_{y\in K} \bar{\mathbb{P}}(X^{\varepsilon,y} \in F) \leq \max_{i=1,\dots,n} \limsup_{\varepsilon\to 0} \varepsilon \ln \sup_{y\in B_{\varepsilon_{x_i}}(x_i)} \bar{\mathbb{P}}(X^{\varepsilon,y} \in F) \leq -J_K^\delta(F).$$

Sending $\delta \to 0$ in the last expression we infer

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \sup_{y\in K} \bar{\mathbb{P}}(X^{\varepsilon,y} \in F) \leq - \inf_{y\in K, f\in F} \mathbb{J}_{y,T}(f).$$

The lower bound follows with analogous arguments. □

2.3.3 The upper bound

We are now in the position to treat the first exit time $\sigma^\varepsilon(x)$ of $X^{\varepsilon,x}$ from D.

Lemma 2.3.1. *Let $\tau > 0$. Then there exists $\rho_0 > 0$ such that, for $0 < \rho < \rho_0$, there exists $s_0 > 0$ such that*

$$\liminf_{\varepsilon\to 0} \varepsilon \ln \inf_{|x|\le\rho} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \le s_0) > -(\bar{V} + \tau),$$

where $\bar{V}$ is defined in (1.2.4).

Proof. Let $\rho_0 > 0$ be small enough such that the inequalities of **Corollary 2.3.1** hold with $\delta = \frac{\tau}{3}$. Choose $x \in D$ such that $|x| \le \rho_0$, $s_x > 0$ and a path $\varphi_1^x \in C([0, s_x], \mathbb{R}^d)$ satisfying $\varphi_1^x(0) = x$, $\varphi_1^x(s_x) = 0$ and

$$\mathbb{J}_{x,s_x}(\varphi_1^x) \le \frac{\tau}{3}.$$

Using **Corollary 2.3.1** and **Proposition 2.3.1** we choose $z \in D^c - \mathrm{cl}D$, $s_z > 0$, $\varphi_2^z \in C([0, s_z], \mathbb{R}^d)$ such that $\varphi_2^z(0) = 0$, $\varphi_2^z(s_z) = z$ and such that

$$\mathbb{J}_{0,s_z}(\varphi_2^z) \le \bar{V} + \frac{\tau}{3}.$$

Let φ_3 be the solution of the differential equation $\dot{\varphi}_3 = -\nabla U(\varphi_3)$ with $\varphi_3(0) = z$. We set $s_0 = s_x + s_z + \delta'$ with $\delta' > 0$ such that $\varphi_3([0, \delta']) \subset D^c - \mathrm{cl}D$. For $x \in D$ such that $|x| \le \rho_0$ we construct

$$\Phi_t^x = \begin{cases} \varphi_1^x(t) & \text{if } 0 \le t \le s_x, \\ \varphi_2^z(t - s_x) & \text{if } s_x \le t \le s_z + s_x, \\ \varphi_3(t - s_z - s_x) & \text{if } s_x + s_z \le t \le s_0. \end{cases}$$

Then we have

$$\mathbb{J}_{x,s_0}(\Phi_t^x) \le \mathbb{J}_{x,s_x}(\varphi_1^x) + \mathbb{J}_{0,s_z}(\varphi_2^z) \le \bar{V} + \frac{2\tau}{3}.$$

Let $\Delta = d(z, \bar{D})$ and consider the open set

$$\mathcal{O} = \bigcup_{|x|\le\rho_0} \{\psi \in \mathbb{D}([0, s_0], \mathbb{R}^d) : d_{J_1}(\psi, \Phi^x) < \frac{\Delta}{2}\}.$$

Φ^x visits z by definition and stays outside of D in the time interval $[s_x + s_z, s_0]$, due to the choice of $z \in D^c - \mathrm{cl}D$ and the continuity of φ_3. By definition of $\mathcal{O}$, every path $\psi \in \mathcal{O}$ exits D before time s_0.We show the statement by contradiction. Fix $\psi \in \mathcal{O}$. Let us suppose that $\psi([0, s_0]) \subset D$. This implies that

$$d(z, \mathrm{cl}(\psi([0, s_0]))) > \Delta. \tag{2.3.14}$$

Since $\psi \in \mathcal{O}$, $d_{J_1}(\psi, \Phi^x) < \frac{\Delta}{2}$. Then there exists an increasing homeomorphism $\lambda : [0, s_0] \longrightarrow [0, s_0]$ such that

$$\sup_{t \in [0,s_0]} |\psi(\lambda(t)) - \Phi^x(t)| < \frac{\Delta}{2}.$$

In particular

$$|\psi(\lambda(s_z + s_x)) - \Phi^x(s_z + s_x)| = |\psi(\lambda(s_z + s_x)) - z| < \frac{\Delta}{2},$$

which contradicts (2.3.14).
Due to **Corollary 2.3.2** we have

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \varepsilon \ln \inf_{|x| \leq \rho_0} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq s_0) &\geq \liminf_{\varepsilon \to 0} \varepsilon \ln \inf_{|x| \leq \rho_0} \bar{\mathbb{P}}(X^{\varepsilon,x} \in \mathcal{O}) \\
&\geq - \sup_{|x| \leq \rho_0} \inf_{\psi \in \mathcal{O}} \mathbb{J}_{x,s_0}(\psi) \\
&\geq - \sup_{|x| \leq \rho_0} \mathbb{J}_{x,s_0}(\Phi^x) \\
&\geq -(\bar{V} + \frac{2\tau}{3}) \\
&> -(\bar{V} + \tau),
\end{aligned}$$

which finishes the proof. □

For fixed $x \in D$, we show next that the probability $X^{\varepsilon,x}$ staying inside D, but without hitting a small neighborhood of 0, is exponentially small. For given $\rho > 0$, such that $\mathrm{cl}B_\rho(0) \subset D$, we define

$$\tau_\rho^\varepsilon(x) := \inf\{t \geq 0 : |X_t^{\varepsilon,x}| \leq \rho \text{ or } X_t^{\varepsilon,x} \in D^c\}.$$

Lemma 2.3.2. *We have*

$$\lim_{t \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in D} \bar{\mathbb{P}}(\tau_\rho^\varepsilon(x) > t) = -\infty$$

Proof. Let us fix $\rho > 0$. For $t \geq 0$, we define the subset of $\mathbb{D}([0,t], \mathbb{R}^d)$

$$\mathcal{G}_t := \Big\{\Phi \in \mathbb{D}([0,t], \mathbb{R}^d) : \Phi(s) \in \mathrm{cl}(D - B_\rho(0)) \quad \text{for all } s \in [0,t]\Big\}.$$

Let us consider the following set,

$$\begin{aligned}
\tilde{\mathcal{G}}_t := \Big\{&\Phi \in \mathbb{D}([0,t], \mathbb{R}^d) : \Phi(s) \in \mathrm{cl}(D - B_\rho(0)) \text{ for all } s \in [0,t] \\
&\text{except possibly in a countable number of points}\Big\}.
\end{aligned}$$

1. We prove that $\tilde{\mathcal{G}}_t$ is closed in $\mathbb{D}([0,t],\mathbb{R}^d)$ with respect to the Skorokhod topology. Let $(\Phi_n)_{n\in\mathbb{N}} \subset \tilde{\mathcal{G}}_t$ such that $d_{J_1}(\Phi_n,\Phi) \to 0$ as $n\to\infty$, for some $\Phi \in \mathbb{D}([0,t],\mathbb{R}^d)$. We denote $(s_k)_{k\in\mathbb{N}}$ the countable set of discontinuity points of Φ. For each $n\in\mathbb{N}$ we denote $(t_k^n)_{k\in\mathbb{N}}$ the countable set such that

$$\Phi_n(s) \in \text{ cl } (D - B_\rho(0)) \quad \text{for all } s\in[0,t]-(t_k^n)_{k\in\mathbb{N}}.$$

For all $s\in[0,t]-\left(\bigcup_{n=1}^\infty (t_k^n)_{k\in\mathbb{N}} \cup (s_k)_{k\in\mathbb{N}}\right)$, due to **Proposition C.2.1**,

$$\Phi_n(s) \to \Phi(s) \quad \text{as } n\to\infty.$$

Since $\text{cl } (D-B_\rho(0))$ is a compact set of $\mathbb{R}^d$, $\Phi(s)\in \text{ cl } (D-B_\rho(0))$, which concludes the proof that $\tilde{\mathcal{G}}_t$ is closed in $(\mathbb{D}([0,T],\mathbb{R}^d),J_1)$.

2. We prove next that $\tilde{\mathcal{G}}_t = \mathcal{G}_t$. The inclusion $\tilde{\mathcal{G}}_t \supset \mathcal{G}_t$ is obvious. Let $\Phi\in\tilde{\mathcal{G}}_t$. If there exists $s\in[0,t]$ such that $\Phi(s)\notin \text{ cl } (D-B_\rho(0))$, since $(\text{cl}D)^c$ and $B_\rho(0)$ are open sets of $\mathbb{R}^d$, by right-continuity of Φ, there exists $\delta>0$ such that

$$\Phi[s,s+\delta) \subset (\text{cl}D)^c \cup B_\rho(0),$$

which violates $\Phi\in\tilde{\mathcal{G}}_t$.

Due to the definition of $\mathcal{G}_t$ and **Corollary 2.3.2**, we have

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon \ln \sup_{x\in D} \bar{\mathbb{P}}(\tau_\rho^\varepsilon(x) > t) &\le \limsup_{\varepsilon\to 0} \varepsilon \ln \sup_{x\in D-\text{cl}(\bar{B}_\rho(0))} \bar{\mathbb{P}}(\tau_\rho^\varepsilon(x) > t) \\
&\le - \inf_{x\in D-\text{cl}(\bar{B}_\rho(0))} \inf_{\psi\in\tilde{\mathcal{G}}_t} \mathbb{J}_{x,t}(\psi) \\
&= - \inf_{x\in D-\bar{B}_\rho(0)} \inf_{\psi\in\mathcal{G}_t} \mathbb{J}_{x,t}(\psi) \\
&= - \inf_{\psi\in\mathcal{G}_t} \mathbb{J}_{\psi(0),t}(\psi).
\end{aligned}$$

Next we show that

$$\lim_{t\to\infty} \inf_{\psi\in\mathcal{G}_t} \mathbb{J}_{\psi(0),t}(\psi) = \infty.$$

Let $(\varphi_t)_{t\ge 0}$ be the dynamical system associated to $\dot{\varphi}_t = -\nabla U(\varphi_t)$. Due to **Condition 1.1.1**, given $x\in D-\text{cl}(B_\rho(0))$, there exists $t_x\ge 0$ such that $\varphi(t_x)\in B_{\frac{\varrho}{2}}(0)$. Define

$$O_x = \varphi^{-1}(B_{\frac{\varrho}{2}}(0)).$$

O_x is an open neighborhood of x in the usual topology of $\mathbb{R}^d$. Choose $x_1,...,x_k\in D-\text{cl}(B_\rho(0))$ such that $\displaystyle\bigcup_{i=1}^k O_{x_i} \supset (D-\text{cl}(B_\rho(0))$ and define $s = t_{x_1}\vee\cdots\vee t_{x_k}$. Before time s, any path that solves $\dot{\varphi}_t = -\nabla U(\varphi_t)$, with initial condition in $D - \mathrm{cl}(B_\rho)(0)$, hits $B_{\frac{\rho}{2}}$. We argue by contradiction. Assume that

$$\lim_{t\to+\infty} \inf_{\psi\in\mathcal{G}_t} \mathbb{J}_{\psi(0),t}(\psi) < \infty.$$

Let us fix $M > 0$ such that, for any $n \in \mathbb{N}$, there exists $\varphi^n \in \mathcal{G}_{ns}$ such that $\mathbb{J}_{\varphi^n(0),ns}(\varphi^n) \leq M$.
For $k = 0, ..., n-1$, let

$$\varphi^{n,k}(t) = \varphi^n(k(s-t)), t \in [0, s].$$

Hence, $\varphi_{n,k} \in \mathcal{G}_s$ and

$$M \geq \mathbb{J}_{\varphi^n(0),ns}(\varphi^n) = \sum_{i=0}^{n-1} \mathbb{J}_{\varphi^n(ks),s}(\varphi^{n,k}) \geq n \min_{0\leq k\leq n-1} \mathbb{J}_{\varphi^{n,k}(0),s}(\varphi^{n,k}).$$

We asserted the existence of a sequence $(\varphi^n)_{n\in\mathbb{N}}$ in $\mathcal{G}_t$ such that

$$\lim_{n\to\infty} \mathbb{J}_{\varphi^n(0),s}(\varphi^n) = 0.$$

The set

$$\{\varphi \in \mathbb{D}([0, s], \mathbb{R}^d) \mid \varphi(0) \in \mathrm{cl}(D - \bar{B}_\rho(0)), \mathbb{J}_{\varphi(0),s}(\varphi) \leq 1\},$$

is closed in the compact set$\{\varphi \in \mathbb{D}([0, s], \mathbb{R}^d) \mid \mathbb{J}_{\varphi(0),s}(\varphi) \leq 1\}$ (since $\mathbb{J}$ is a good rate function) and therefore it is compact for the Skorokhod topology. Hence, $(\varphi^n)_{n\in\mathbb{N}}$ has a limit point in $\mathcal{G}_s$ which we call $\bar{\varphi}$.
Since $\mathbb{J}_{\varphi(0),s} = \inf_{x\in\mathbb{R}^d} \mathbb{J}_{x,s}$ is lower semicontinuous, it follows that $\mathbb{J}_{\bar{\varphi}(0),s}(\bar{\varphi}) = 0$, which means that $\bar{\varphi}$ solves $\dot{\varphi}_t = -\nabla U(\varphi_t)$ with $\bar{\varphi}(0) \in D - \mathrm{cl}(B_\rho(0))$. By what it was said before, $\bar{\varphi}$ reaches $B_{\frac{\varrho}{2}}(0)$ before time s, which contradicts $\bar{\varphi} \in \mathcal{G}_s$. $\square$

Theorem 2.3.2. *For $x \in D$ and $\delta > 0$ we have*

$$\liminf_{\varepsilon\to 0} \varepsilon \ln \bar{\mathbb{P}}(\sigma^\varepsilon(x) < e^{\frac{\bar{V}+\delta}{\varepsilon}}) = 1.$$

Proof. We prove the following claim.

Claim 2.3.1. For any $\delta > 0$ there exists $T > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, we have

$$\inf_{x\in D} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq T) \geq e^{-\frac{\bar{V}+\frac{\delta}{2}}{\varepsilon}}.$$

We start with the observation that by **Lemma 2.3.1** for every $\delta > 0$ there exist $t_0 > 0$ and $\rho > 0$ such that

$$\liminf_{\varepsilon\to 0} \varepsilon \ln \inf_{|x|\leq\rho} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq t_0) > -(\bar{V} + \frac{\delta}{4}).$$

Applying **Lemma 2.3.2** for the fixed value ρ, there exists a time $t_1 > 0$ such that

$$\limsup_{\varepsilon \to 0^+} \varepsilon \ln \sup_{x \in D} \bar{\mathbb{P}}(\tau_\rho^\varepsilon(x) > t_1) < 0.$$

This implies, for any $r > 0$, the existence of $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,

$$\varepsilon \ln \sup_{x \in D} \bar{\mathbb{P}}(\tau_\rho^\varepsilon(x) > t_1) < -r.$$

In addition, we fix $\varepsilon_0 > 0$ small enough such that $\varepsilon < \varepsilon_0$ implies $1 - e^{-\frac{r}{\varepsilon}} > e^{-\frac{\delta}{4\varepsilon}}$.
Since $\{\tau_\rho^\varepsilon(x) < \sigma^\varepsilon(x)\} = \{X^{\varepsilon,x}_{\tau_\rho^\varepsilon(x)} \in \mathrm{cl}(B_\rho(0))\}$ it follows on this event

$$\sigma^\varepsilon(x) = \tau_\rho^\varepsilon(x) + \sigma^\varepsilon(X^{\varepsilon,x}_{\tau_\rho^\varepsilon(x)}) \circ \Theta_{\tau_\rho^\varepsilon(x)},$$

where Θ_s is the shift by time s on the path space $\mathbb{D}([0,\infty), \mathbb{R}^d)$. Using the homogeneous strong Markov property of $X^{\varepsilon,x}$ (see **Proposition 1.1.1**) we obtain, for fixed $\varepsilon < \varepsilon_0$ and $x \in D$,

$$\begin{aligned}
\bar{\mathbb{P}}(\sigma^\varepsilon(x) \le t_0 + t_1) &\ge \bar{\mathbb{P}}\Big(\tau_\rho^\varepsilon(x) \le t_1 \text{ and } \sigma^\varepsilon(X^{\varepsilon,x}_{\tau_\rho^\varepsilon(x)}) \le t_0\Big) \\
&= \bar{\mathbb{P}}\Big(\tau_\rho^\varepsilon(x) \le t_1\Big)\bar{\mathbb{P}}\Big(\sigma^\varepsilon(X^{\varepsilon,x}_{\tau_\rho^\varepsilon(x)}) \le t_0 | \tau_\rho^\varepsilon(x) \le t_1\Big) \\
&\ge \inf_{y \in D} \bar{\mathbb{P}}(\tau_\rho^\varepsilon(y) \le t_1) \inf_{|x| \le \rho} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \le t_0) \\
&\ge e^{-\frac{\bar{V} + \frac{\delta}{4}}{\varepsilon}} e^{-\frac{\delta}{4\varepsilon}} \\
&\ge e^{-\frac{\bar{V} + \frac{\delta}{4}}{\varepsilon}} (1 - e^{-\frac{r}{\varepsilon}}) \\
&= e^{-\frac{\bar{V} + \frac{\delta}{2}}{\varepsilon}}.
\end{aligned}$$

Setting $T = t_0 + t_1$ we have proved the claim.

We set $q^\varepsilon = \inf_{x \in D} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \le T)$. **Claim 2.3.1** yields $q^\varepsilon > 0$ for $\varepsilon < \varepsilon_0$. For any $k \in \mathbb{N}$ and $x \in D$ we consider the family of events $\{\sigma^\varepsilon(x) > kT\}$ for which we derive the following recursion

$$\begin{aligned}
\bar{\mathbb{P}}\Big(\sigma^\varepsilon(x) > (k+1)T\Big) &= \Big(1 - \bar{\mathbb{P}}\Big(\sigma^\varepsilon(x) \le (k+1)T | \sigma^\varepsilon(x) > kT\Big)\Big)\bar{\mathbb{P}}\Big(\sigma^\varepsilon(x) > kT\Big) \\
&\le (1 - q^\varepsilon)\bar{\mathbb{P}}\Big(\sigma^\varepsilon(x) > kT\Big), \quad k \in \mathbb{N}.
\end{aligned}$$

Solving the recursion above in $k \in \mathbb{N}$ we obtain, for $\varepsilon < \varepsilon_0$,

$$\sup_{x \in D} \bar{\mathbb{P}}(\sigma^\varepsilon(x) > kT) \le (1 - q^\varepsilon)^k.$$

This implies the following bound

$$\begin{aligned}\sup_{x\in D}\bar{\mathbb{E}}[\sigma^\varepsilon(x)] &= \sup_{x\in D} T\int_0^\infty \bar{\mathbb{P}}(\sigma^\varepsilon(x) > Ts)ds \\ &\le T\sup_{x\in D}\sum_{k=0}^{+\infty}\bar{\mathbb{P}}(\sigma^\varepsilon(x) > kT) \\ &\le T\sum_{k=0}^{+\infty}(1-q^\varepsilon)^k \\ &= \frac{T}{q^\varepsilon}.\end{aligned}$$

Since we have $q^\varepsilon \ge e^{-\frac{\bar{V}+\frac{\delta}{2}}{\varepsilon}}$ for $\varepsilon < \varepsilon_0$ we obtain

$$\sup_{x\in D}\mathbb{E}[\sigma^\varepsilon(x)] \le Te^{\frac{\bar{V}+\frac{\delta}{2}}{\varepsilon}}.$$

Chebyshev's inequality implies, for all $x \in D$ and $\varepsilon < \varepsilon_0$,

$$\bar{\mathbb{P}}(\sigma^\varepsilon(x) \ge e^{\frac{\bar{V}+\delta}{\varepsilon}}) \le e^{-\frac{\bar{V}+\delta}{\varepsilon}}\mathbb{E}[\sigma^\varepsilon(x)] \le e^{-\frac{\delta}{2\varepsilon}}.$$

Sending $\varepsilon \to 0$ the lower bound is proved. □

2.3.4 The lower bound

Let $x \in D$ and $\rho > 0$ such that $\mathrm{cl}(B_\rho(0)) \subset D$. We keep the notation of the last subsection.

Lemma 2.3.3. *For any $x \in D$ and $\rho > 0$ such that $\bar{B}_\rho(0) \subset D$ we have*

$$\lim_{\varepsilon\to 0}\bar{\mathbb{P}}(X^{\varepsilon,x}_{\tau^\varepsilon_\rho(x)} \in \bar{B}_\rho(0)) = 1.$$

Proof. We fix $\rho > 0$ and $x \in D - \mathrm{cl}(B_\rho(0))$. Otherwise the result is trivial.
Let $u(.;x)$ be the solution of

$$\begin{cases}\dot{u}(t;x) &= -\nabla U(u(t;x)), \quad t \ge 0 \\ u(0;x) &= x,\end{cases} \tag{2.3.15}$$

and for all $t \ge 0$,

$$w(t;x) = x - \int_0^t \nabla U(w(s;x))ds - \int_0^t\int_{\mathbb{R}^d} z\nu(dz)ds. \tag{2.3.16}$$

Let us define

$$t_0 := \inf\{t \ge 0 : u(t;x) \in \bar{B}_{\frac{\rho}{3}}(0)\}.$$

Due to **Condition 1.1.1**-(ii) $t_0 < \infty$ and

$$\rho_0 := \rho \wedge d(u([0,t_0];x), D^c) > 0.$$

We fix $t_1 \geq \frac{\rho_0}{3\int_{\mathbb{R}^d}|z|\nu(dz)}$ and $s = t_0 + t_1$.
Due to **Condition 1.1.1**, the definition of the time $s = t_0 + t_1$, (2.3.8) and (2.3.10) there exist $\varepsilon_0 > 0$ and $c_1(\rho_0), c_2(\rho) > 0$ such that, for $\varepsilon < \varepsilon_0$ we have

$$\begin{aligned}
\bar{\mathbb{P}}\Big(X^{\varepsilon,x}_{\tau^x_\rho} \notin \bar{B}_{\rho_0}(0)\Big) &\leq \bar{\mathbb{P}}\Big(\sup_{t\in[0,s]}|X^{\varepsilon,x}_t - w(t;x)| > \frac{\rho_0}{3}\Big) \\
&\quad + \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq s}|w(t;x) - u(t;x)| > \frac{\rho_0}{3}\Big) \\
&\quad + \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq s}|u(t;x)| > \frac{\rho_0}{3}\Big) \\
&\leq \bar{\mathbb{P}}\Big(\sup_{n\in\mathbb{N}}|X^{\varepsilon,x}_{\varepsilon T_n} - w(\varepsilon T_n;x)| > \frac{\rho_0}{3}\Big) \\
&\leq c_1(\rho_0)e^{-\frac{c_2(\rho_0)}{\varepsilon^\alpha}},
\end{aligned}$$

where $(\varepsilon T_n)_{n\in\mathbb{N}}$ is the family of the jump times defined in (2.3.3). Sending $\varepsilon \to 0$ we conclude the result. □

Lemma 2.3.4. *For $\rho > 0$ and $c > 0$, there exists $T(\rho) > 0$ such that $0 \leq t \leq T(\rho)$ implies*

$$\limsup_{\varepsilon\to 0}\varepsilon\ln\sup_{x\in D}\bar{\mathbb{P}}(\sup_{0\leq t\leq T(\rho)}|X^{\varepsilon,x}_t - x| \geq \rho) < -c.$$

Proof. Fix $\rho > 0$, $\varepsilon > 0$ and $x \in D$. Let $u(.;x)$ the solution of the initial value problem (2.3.15). Since **Condition 1.1.1**-(ii) holds, the image of $u(.;x)$ is contained in D. Since $-\nabla U$ is continuous, let $K = K_D > 0$ such that

$$|-\nabla U(x)| \leq K \quad \text{for all } x \in D.$$

We define

$$T(\rho) := \frac{\rho}{2(K + \int_{\mathbb{R}^d}|z|\nu(dz))} + 1.$$

Recalling $w(;x)$ defined by (2.3.16), (2.3.8) and (2.3.10) imply that there exist $\varepsilon_0 > 0$ and $c_1(\rho), c_2(\rho) > 0$ such that, for every $\varepsilon < \varepsilon_0$, we have

$$\begin{aligned}
\bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T(\rho)}|X^{\varepsilon,x}_t - x| \geq \rho\Big) &\leq \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T(\rho)}|X^{\varepsilon,x}_t - w(t;x)| \geq \frac{\rho}{2}\Big) \\
&\quad + \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T(\rho)}|w(t;x) - x| \geq \frac{\rho}{2}\Big) \\
&\leq \bar{\mathbb{P}}\Big(\sup_{n\in\mathbb{N}}|X_{\varepsilon T_n} - w(\varepsilon T_n;x)| > \frac{\rho}{2}\Big) \\
&\leq c_1(\rho)e^{-c_2(\rho)\varepsilon^{-\alpha}},
\end{aligned}$$

for $\varepsilon < \varepsilon_0$. Here $(\varepsilon T_n)_{n\in\mathbb{N}}$ is the family of the jumping times defined in (2.3.3). Sending $\varepsilon \to 0$, we obtain the desired result. $\square$

Lemma 2.3.5. *Let $F \supset D^c$ closed. Then*

$$\lim_{\rho\to 0}\limsup_{\varepsilon\to 0}\varepsilon \ln \sup_{|x|\leq 2\rho} \bar{\mathbb{P}}(X^{\varepsilon,x}_{\tau^x_\rho} \in F) \leq -\inf_{z\in F} V(0,z).$$

Proof. Fix $\delta > 0$ and $V_F(\delta) := \min\{(\inf_{z\in F} V(0,z) - \delta), \frac{1}{\delta}\}$. By definition of V, we conclude

$$V(x,z) \leq V(x,y) + V(y,z) \;\; \forall x,y,z \in \mathbb{R}^d.$$

Using **Corollary 2.3.1**, for $0 < \rho < \rho_0$ with $\rho_0 > 0$ small enough,

$$\inf_{z\in F, |y|\leq 2\rho} V(y,z) \geq \inf_{z\in F} V(0,z) - \sup_{|y|\leq 2\rho} V(0,y) \geq V_F(\delta).$$

Using **Lemma 2.3.4**, we choose $\tilde{T} > 0$ such that for any $0 < \rho < \rho_0$

$$\limsup_{\varepsilon\to 0}\varepsilon \ln \sup_{|y|\leq 2\rho} \bar{\mathbb{P}}(\tau^\varepsilon_\rho(y) > \tilde{T}) \leq \limsup_{\varepsilon\to 0}\varepsilon \ln \sup_{\rho\leq|y|\leq 2\rho} \bar{\mathbb{P}}(\tau^\varepsilon_\rho(y) > \tilde{T}) < -V_F(\delta).$$

We consider the following subset of $\mathbb{D}([0,\tilde{T}],\mathbb{R}^d)$,

$$\mathcal{A} := \{\varphi \in \mathbb{D}([0,\tilde{T}],\mathbb{R}^d) \mid \varphi(s) \in F \quad \text{for some } s \in [0,\tilde{T}]\}.$$

We argue that $\mathcal{A}$ is a closed set of $\mathbb{D}([0,\tilde{T}],\mathbb{R}^d)$ for the Skorokhod topology. Let $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathcal{A}$ and $\varphi \in \mathbb{D}([0,\tilde{T}],\mathbb{R}^d)$ such that $d_{J_1}(\varphi_n,\varphi) \to 0$ as $n \to \infty$.
For every $n \in \mathbb{N}$, let $s_n \in [0,\tilde{T}]$ such that $\varphi_n(s_n) \in F$. By right continuity of φ_n, there exists $\delta_n > 0$ such that $\varphi_n([s_n, s_n+\delta_n)) \subset F$. For every $n \in \mathbb{N}$, we denote $I_n := [s_n, s_n+\delta_n)$. Due to **Proposition C.2.1**, for every $n \in \mathbb{N}$ let $(t^k_n)_{k\in\mathbb{N}}$ be the set of discontinuities of φ in I_n. Therefore, **Proposition C.2.2** implies that

$$\varphi_n(r) \to \varphi(r), \qquad \text{for all } r \in \bigcup_{n\in\mathbb{N}}(I_n - (t^k_n)_{k\in\mathbb{N}}).$$

Since F is a closed subset of $\mathbb{R}^d$, $\varphi(r) \in F$, for all $r \in \bigcup_{n\in\mathbb{N}}(I_n - (t^k_n)_{k\in\mathbb{N}})$. This proves that $\varphi \in \mathcal{A}$ and that $\mathcal{A}$ is closed in $\mathbb{D}([0,\tilde{T}],\mathbb{R}^d)$ for the Skorokhod topology.

Theorem 2.3.1 implies that there exists $\rho_0 > 0$ such that, for $0 < \rho < \rho_0$,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon \ln \sup_{|y|\leq 2\rho} \bar{\mathbb{P}}(X^{\varepsilon,y} \in \mathcal{A}) &\leq -\inf_{|y|\leq 2\rho}\inf_{\varphi\in\mathcal{A}} \mathbb{J}_{y,T}(\varphi)\\ &\leq -\inf_{|y|\leq 2\rho, z\in F} V(y,z)\\ &\leq -V_F(\delta).\end{aligned}$$

In conclusion,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\ln\sup_{|x|\le 2\rho}\bar{\mathbb{P}}(X^{\varepsilon,x}_{\tau^x_\rho}\in F)&\le\limsup_{\varepsilon\to 0}\varepsilon\ln\sup_{|x|\le 2\rho}\bar{\mathbb{P}}(\tau^\varepsilon_\rho(x)<\infty)\\&\le\limsup_{\varepsilon\to 0}\varepsilon\ln\sup_{|x|\le 2\rho}\bar{\mathbb{P}}(\{\tau^\varepsilon_\rho(x)>\tilde{T}\}\cup\{\tau^\varepsilon_\rho(x)\le\tilde{T}\})\\&\le\limsup_{\varepsilon\to 0}\varepsilon\ln\Big(\sup_{|y|\le 2\rho}\bar{\mathbb{P}}(\tau^\varepsilon_\rho(y)>\tilde{T})+\sup_{|y|\le 2\rho}\bar{\mathbb{P}}(X^{\varepsilon,y}\in\mathcal{A})\Big)\\&\le -V_F(\delta),\end{aligned}$$

and the result follows sending $\delta\to 0$. □

Theorem 2.3.3. *Let $\delta>0, x\in D$. We have*

$$\lim_{\varepsilon\to 0}\mathbb{P}(\sigma^\varepsilon(x)\le e^{\frac{\bar{V}-\delta}{\varepsilon}})=0.$$

Proof. Choose $\rho>0$ such that $\mathrm{cl}(B_\rho(0))\subset D$. Define recursively, for $x\in D$ and $k\in\mathbb{N}$,

$$\begin{aligned}\zeta^x_0&:=0,\\\tau^x_k&:=\inf\{t\ge\zeta^x_k\mid X^{\varepsilon,x}_t\in\mathrm{cl}(B_\rho(0))\cup D^c\},\\\zeta^x_{k+1}&:=\begin{cases}\infty & \text{if } X^{\varepsilon,x}_{\tau^x_k}\in D^c,\\\inf\{t\ge\tau^x_k\mid X^{\varepsilon,x}_t\in(\mathrm{cl}(B_\rho(0)))^c\} & \text{if } X^{\varepsilon,x}_{\tau^x_k}\in\mathrm{cl}(B_\rho(0)).\end{cases}\end{aligned}$$

Due to the way $(\zeta^x_k)_{k\in\mathbb{N}}$ and $(\tau^x_k)_{k\in\mathbb{N}}$ were defined we have,for all $k\in\mathbb{N}$ $\bar{\mathbb{P}}$-a.s.

$$\zeta^x_k\le\tau^x_k\le\zeta^x_{k+1}.$$

We discuss three different sample paths of $(X^{\varepsilon,x}_t)_{t\ge 0}$ in order to give some intuition to the reader about the way $(\zeta^x_k)_{k\in\mathbb{N}}$ and $(\tau^x_k)_{k\in\mathbb{N}}$ were defined.

1. In the first sample path, $X^{\varepsilon,x}_{\tau^x_k}\in\mathrm{cl}(B_\rho(0))$. Then after some finite number of jumps the jump diffusion exits from $\mathrm{cl}(B_\rho(0))$ and enters $D-\mathrm{cl}(B_\rho(0))$. Then $X^{\varepsilon,x}_{\zeta^x_{k+1}}\in D-\mathrm{cl}(B_\rho(0))$. After some other finite number of jumps $X^{\varepsilon,x}$ exits the domain D and enters in D^c. Then $X^{\varepsilon,x}_{\tau^x_{k+1}}\in D^c$ and $\zeta^x_{k+2}=\infty$ and consequently $\tau^x_{k+2}=\zeta^x_{k+2}$. By definition, $\zeta^x_{k+3}=\infty$ and consequently $\tau^x_m=\zeta^x_m=\infty$ for all $m\ge k+4$.

2. In the second sample path, $X^{\varepsilon,x}_{\tau^x_k}\in\mathrm{cl}(B_\rho(0))$. After some finite number of jumps $X^{\varepsilon,x}$ exits D without entering $D-\mathrm{cl}(B_\rho(0))$. Hence, $X^{\varepsilon,x}_{\zeta^x_{k+1}}\in D^c$ and $\tau^x_{k+1}=\zeta^x_{k+1}$. By definition, $\zeta^x_{k+2}=\infty$ and therefore $\tau^x_m=\zeta^x_m=\infty$ for all $m\ge 3$.

3. In the third sample, the initial position of the jump diffusion $x\in D-\mathrm{cl}(B_\rho(0))$ and after a finite number of steps without entering the ball $\mathrm{cl}(B_\rho(0))$, $X^{\varepsilon,x}$ exits the domain D. Then $X^{\varepsilon,x}_{\tau^x_0}\in D^c$ and therefore $\zeta^x_1=\infty$ and therefore $\tau^x_1=\zeta^x_1=\infty$ and therefore $\tau^x_m=\zeta^x_m=\infty$ for all $m\ge 2$.

The facts that $(\tau_k^x)_{k\in\mathbb{N}}$ is a sequence of stopping times and, for every $\varepsilon > 0$, $(X^{\varepsilon,x})_{t\geq 0}$ is strong Markov (**Proposition 1.1.1**) imply that $(X^{\varepsilon,x}_{\tau_k^x})$ is a Markov chain, with the convention $X^{\varepsilon,x}_{\tau_k^x} := X^{\varepsilon,x}_{\sigma^\varepsilon(x)}$ if $\tau_k^x = \infty$.
We observe that, for every $\varepsilon > 0$ and $x \in D$, $\sigma^\varepsilon(x) = \tau_k^x$ for some $k \in \mathbb{N}$. Fix $\delta > 0$. Using **Lemma 2.3.5**, there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, $k \in \mathbb{N}$, we have

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \sup_{|x|\leq 2\rho} \bar{\mathbb{P}}(X^{\varepsilon,x}_{\tau_\rho^x} \in D^c) \leq -\bar{V} + \frac{\delta}{2}.$$

Fix now $\rho < \rho_0$. Choose $\tilde{T} = \tilde{T}(\rho)$ according to **Lemma 2.3.4**. Then due to the strong Markov property (**Proposition 1.1.1**) there exists $\rho_0 > 0$ such that for $\rho \leq \rho_0$ we have

$$\begin{aligned}\sup_{x\in D} \bar{\mathbb{P}}(\sigma^\varepsilon(x) = \tau_x^k) &\leq \sup_{|y|\leq 2\rho} \bar{\mathbb{P}}(X^{\varepsilon,y}_{\tau_\rho^y} \in D^c) \\ &\leq e^{-\frac{\bar{V}-\frac{\delta}{2}}{\varepsilon}}\end{aligned}$$

and

$$\begin{aligned}\sup_{x\in D} \bar{\mathbb{P}}(\zeta_k^x - \tau_{k-1}^x \leq \tilde{T}) &\leq \sup_{|x|\leq\rho} \bar{\mathbb{P}}(\sup_{0\leq t\leq\tilde{T}} |X_t^{\varepsilon,x} - x| \geq \rho) \\ &\leq \sup_{x\in D} \bar{\mathbb{P}}(\sup_{0\leq t\leq\tilde{T}} |X_t^{\varepsilon,x} - x| \geq \rho) \\ &\leq e^{-\frac{\bar{V}-\frac{\delta}{2}}{\varepsilon}}.\end{aligned}$$

Let $n \in \mathbb{N}$. For any $x \in D$, $\varepsilon > 0$ and $k \in \mathbb{N}$ we prove

$$\{\sigma^\varepsilon(x) \geq \tau_x^k\} \cap \bigcap_{m=1}^{k} \{\zeta_m^k - \tau_{m-1}^x \geq \tilde{T}\} \subset \{\sigma^\varepsilon(x) \geq K\tilde{T}\}$$

The following inclusion follows from noting that if $\sigma^\varepsilon(x) \geq \tau_k^x$ and $\zeta_m^x - \tau_{m-1}^x \geq \tilde{T}$ for all $m = 1, \ldots, k$, we have

$$\tau_k^x = \sum_{m=1}^{k} (\tau_{m+1}^x - \tau_m^x) + \tau_0^x \geq \sum_{m=1}^{k} (\zeta_m^x - \tau_m^x) \geq k\tilde{T}.$$

This implies

$$\{\sigma^\varepsilon(x) \leq k\tilde{T}\} \subset \{\sigma^\varepsilon(x) = \tau_0^x\} \cup \bigcup_{m=1}^{k} \{\sigma^\varepsilon(x) = \tau_m^x\} \cup \{\zeta_m^x - \tau_{m-1}^x \leq \tilde{T}\}.$$

Hence, for any $k \in \mathbb{N}$ and $x \in D$,

$$\begin{aligned}\bar{\mathbb{P}}(\sigma^\varepsilon(x) \leq k\tilde{T}) &\leq \bar{\mathbb{P}}(\sigma^\varepsilon(x) = \tau_0^x) + \sum_{m=1}^{k} \Big(\bar{\mathbb{P}}(\sigma^\varepsilon(x)) = \tau_m^x) + \bar{\mathbb{P}}(\zeta_m^x - \zeta_{m-1}^x \leq \tilde{T})\Big) \\ &\leq \bar{\mathbb{P}}(\sigma^\varepsilon(x)) = \tau_x^0) + 2ke^{-\frac{\bar{V}-\frac{\delta}{2}}{\varepsilon}}.\end{aligned}$$

Set $k := k(\varepsilon) = [\frac{1}{\tilde{T}} e^{\frac{\bar{V}-\delta}{\varepsilon}}] + 1$.
Due to **Lemma 2.3.3**, we have, for all $x \in D$,

$$\begin{aligned}\bar{\mathbb{P}}(\sigma^\varepsilon(x) \le e^{\frac{\bar{V}-\delta}{\varepsilon}}) &\le \bar{\mathbb{P}}(\sigma^\varepsilon(x) \le k\tilde{T}) \\ &\le \bar{\mathbb{P}}(X^{\varepsilon,x}_{\tau^x_\rho} \notin \bar{B}_\rho(0)) + \frac{4}{\tilde{T}} e^{-\frac{\delta}{2\varepsilon}} \to 0 \quad \text{as} \varepsilon \to 0.\end{aligned}$$

Chebyshev's inequality implies that, for some $c(\tilde{T}) > 0$, we have

$$\begin{aligned}\bar{\mathbb{E}}[\sigma^\varepsilon(x)] &\ge e^{\frac{\bar{V}+\delta}{\varepsilon}} \bar{\mathbb{P}}(\sigma^\varepsilon(x) \le e^{\frac{\bar{V}-\delta}{\varepsilon}}) \\ &\ge c(\tilde{T}) e^{\frac{\bar{V}+\frac{\delta}{2}}{\varepsilon}},\end{aligned}$$

We show the $\bar{V} = 0$ in what follows. Let $\delta > 0$ and $x \in D$. Choose $\rho > 0$ such that $\bar{B}_\rho(0) \subset D$. Assume $c > 0$ and by **Lemma 2.3.3** and **Lemma 2.3.5** combined with the strong Markov property, we can choose $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$\bar{\mathbb{P}}(\sigma^\varepsilon(x) > e^{-\frac{\delta}{\varepsilon}}) \ge \bar{\mathbb{P}}(X^{\varepsilon,x}_{\tau^x_\rho} \in \bar{B}_\rho(0)) \inf_{|y| \le 2\rho} \bar{\mathbb{P}}(\sup_{0 \le t \le T(c,\rho)} |X^{\varepsilon,y} - y| \le \rho) \to 1,$$

as $\varepsilon \to 0$, which concludes the proof. $\square$

Chapter 3

A moderate deviations principle and the first exit times asymptotics for subexponential jump diffusions

3.1 Preliminaries and a sufficient condition for a moderate deviations principle

Let us fix a finite time $T > 0$ and $(X^{\varepsilon,x})_{\varepsilon>0}$ the solution of (1.1.4) in the sense of **Definition 1.1.1**. Assume that the measure ν is of the form (1.1.5) for some $\alpha \in (0,1)$. When there is no need to stress the dependence of $(X^{\varepsilon,x})_{t\in[0,T]}$ on the initial condition $x \in \mathbb{R}^d$ we write $(X^{\varepsilon})_{t\in[0,T]}$. We fix a bounded domain $D \subset \mathbb{R}^d$ satisfying **Condition 1.1.1**.
Fix a measurable function $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with the asymptotic behavior, as $\varepsilon \to 0$,

$$a(\varepsilon) \to 0 \text{ and } b(\varepsilon) := \frac{\varepsilon}{a(\varepsilon)} \to 0.$$

We introduce the following sets of functions. We consider the set of controls

$$\bar{\mathcal{A}} := \{\psi : [0,T] \times \mathbb{R}^d \times \bar{\mathbb{M}} \longrightarrow \mathbb{R} \mid \varphi \text{ is } (\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))\text{measurable } \},$$

$$\mathcal{S}^M_{+,\varepsilon} := \Big\{\varphi : [0,T] \times \mathbb{R}^d \to [0,+\infty) : \mathfrak{L}_T(\varphi) \le M a^2(\varepsilon)\Big\} \text{ and}$$
$$\mathcal{S}^M_{\varepsilon} := \Big\{\psi : \mathbb{R}^d \times [0,T] \longrightarrow \mathbb{R} : \psi = \frac{\varphi - 1}{a(\varepsilon)}, \varphi \in \mathcal{S}^M_{+,\varepsilon}\Big\}, \tag{3.1.1}$$

where $\mathfrak{L}_T$ is defined in (1.1.8).

We consider an exhaustion $(K_n)_{n\in\mathbb{N}}$ of compact sets of $\mathbb{R}^d$. For every $n \in \mathbb{N}$, we define the positive n-bounded controls

$$\bar{\mathcal{A}}^+_{b,n} := \Big\{\varphi \in \bar{\mathcal{A}}^+ \mid \text{ for all } (t,x,\bar{m}) \in [0,T] \times \mathbb{R}^d \times \bar{\mathbb{M}} :$$
$$\frac{1}{n} \le \varphi(t,x,\bar{m}) \le n, \text{ if } x \in K_n; \quad \varphi(t,x,\bar{m}) = 1, \text{ if } x \in K_n^c.\Big\},$$

and set

$$\bar{\mathcal{A}}_b := \bigcup_{n=1}^{\infty} \bar{\mathcal{A}}_{b,n}.$$

For every $\varepsilon > 0$ and $M > 0$, let

$$\begin{aligned} \mathcal{U}^M_{+,\varepsilon} &:= \{\varphi \in \bar{\mathcal{A}}^+_b \mid \varphi(.,.,\bar{m}) \in \mathcal{S}^M_{+,\varepsilon} \ \bar{\mathbb{P}} - \text{ a.s. } \} \text{ and} \\ \mathcal{U}^M_{\varepsilon} &:= \{\psi \in \bar{\mathcal{A}} \mid \psi(.,.,\bar{m}) \in \mathcal{S}^M_{\varepsilon} \ \bar{\mathbb{P}} - \text{ a.s. } \}. \end{aligned} \tag{3.1.2}$$

We write $\nu_T = ds \otimes \nu$, where ds stands for the Lebesgue measure on $[0, T]$.
The norm in the Hilbert space $L^2(\nu_T)$ will be denoted by $||.||_2$ and for every $R > 0$, $B^2(R)$ will denote the ball of radius R in $L^2(\nu_T)$ centered in 0. Due to **Theorem A.1.1** the ball $B^2(R)$ is a compact metric space when equiped with the weak topology of $L^2(\nu_T)$.
A family of functions $(\psi^\varepsilon)_{\varepsilon>0} \subset \bar{\mathcal{A}}$ satisfying for some $M > 0$, the uniform bound,

$$\sup_{\varepsilon>0} ||\psi||_2^2 \leq M,$$

is regarded as a collection of $B^2(M)$- random variables, where $B^2(M)$ is equipped with the weak topology on the Hilbert space $L^2(\nu_T)$. Since $B^2(M)$ is weakly compact such collection of random variables is automatically tight.

We state the sufficient condition obtained in *Budhiraja et al. (2015)* for a moderate deviations principle.

Condition 3.1.1 (MDP condition). *Let $\mathcal{D}$ be a Polish space and for every $\varepsilon > 0$ $\mathcal{G}^\varepsilon : \mathbb{M} \longrightarrow \mathcal{D}$, $\mathcal{G}^0 : L^2(\nu_T) \longrightarrow \mathcal{D}$ measurable maps satisfying the following conditions where $\kappa_2 > 0$ is the constant of* ***Lemma D.7.1****.*

(i) For any $M > 0$, let $g \in B^2(M)$ and $(g_\varepsilon)_{\varepsilon>0} \subset B^2(M)$ such that $g_\varepsilon \to g$ weakly in the $L^2(\nu_T)$-sense as $\varepsilon \to 0$. Then $\mathcal{G}^0(g)$ is a limit point of $(\mathcal{G}^0(g_\varepsilon))_{\varepsilon>0}$ as $\varepsilon \to 0$.

(ii) For any $M > 0$ and $\varepsilon > 0$, let $\varphi, \varphi_\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$ and set $\psi_\varepsilon = \frac{\varphi_\varepsilon - 1}{a(\varepsilon)}$. Assume for some $\beta \in (0, 1]$ we have the convergence in law $\psi_\varepsilon \mathbf{1}_{\{|\psi_\varepsilon| \leq \beta/a(\varepsilon)\}} \Rightarrow \psi$ in $B_2(\sqrt{Mk_2(1)})$ as $\varepsilon \to 0$. Then

$$\mathcal{G}^0(\psi) \textit{ is a limit point in law of } \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}) \quad \textit{as } \varepsilon \to 0.$$

Let $\mathcal{D}$ be a Polish space. For given $\eta \in \mathcal{D}$, we write

$$\mathbb{T}_\eta := \mathbb{T}_\eta[\mathcal{G}^0] := \{\psi \in L^2(\nu_T) \mid \eta = \mathcal{G}^0(\psi)\}$$

and define

$$\mathbb{I} : \mathcal{D} \longrightarrow [0, \infty]$$

by

$$\mathbb{I}(\eta) = \inf_{\psi \in \mathbb{T}_\eta} \frac{1}{2} ||\psi||_2^2. \tag{3.1.3}$$

The following result is proved in **Section D.7** of the **Appendix**.

Theorem 3.1.1. *Let $\mathcal{D}$ be a Polish space and for every $\varepsilon > 0$ $\mathcal{G}^\varepsilon$ and $\mathcal{G}^0$ measurable maps satisfying* ***Condition 3.1.1****. Then $\mathbb{I}$ defined by (3.1.3) is a good rate function and $(Z^\varepsilon)_{\varepsilon>0}$ defined by $Z^\varepsilon = \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}}), \quad \varepsilon > 0$ satisfies a large deviations principle in $\mathcal{D}$ with speed $b(\varepsilon)$ and good rate function $\mathbb{I}$ as $\varepsilon \to 0$.*

Remark 3.1.2.

1. Fix $T > 0$. Then the family of measurable maps of **Condition 3.1.1** is given by

$$\mathcal{G}^\varepsilon : \mathbb{M} \longrightarrow \mathbb{D}([0,T], \mathbb{R}^d),$$
$$\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}}) := (Y^\varepsilon)_{t\in[0,T]},$$

with $(Y^\varepsilon)_{\varepsilon>0}$ defined in (1.3.1). We define the measurable map

$$\mathcal{G}^0 : L^2(\nu_T) \longrightarrow C([0,T], \mathbb{R}^d),$$
$$\mathcal{G}^0(\psi) = \eta,$$

where $\eta \in C([0,T], \mathbb{R}^d)$ is the unique solution of

$$\eta(t) = -\int_0^t \nabla^2 U(X_s^0)\eta(s)ds + \int_0^t \int_{\mathbb{R}^d} z\psi(s,z)\nu(dz)ds, \quad t \in [0,T].$$

2. Fixed $T > 0$ and $x \in \mathbb{R}^d$. Then the family of measurable maps of **Condition 3.1.1** is given by

$$\mathcal{G}^\varepsilon : \mathbb{M} \longrightarrow \mathbb{D}([0,T], \mathbb{R}^d),$$
$$\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}}) := (X^\varepsilon)_{t\in[0,T]},$$

where, for every $\varepsilon > 0$, $(X_t^\varepsilon)_{t\in[0,T]}$ is the solution of (1.1.4) in the sense of **Definition 1.1.1**.
We define the measurable map

$$\mathcal{G}^0 : L^2(\nu_T) \longrightarrow C([0,T], \mathbb{R}^d),$$
$$\mathcal{G}^0(\psi) = X^0,$$

where X^0 is the unique solution of

$$X_t^0 = x - \int_0^t \nabla U(X_s^0)ds, \quad t \in [0,T].$$

3.2 A moderate deviations principle

We assume the setup discussed in the beginning of the last section.

Preparations

Lemma 3.2.1. *Fix $M > 0$. Then there exists $\tau > 0$ such that for any Borel measurable $I \subset [0,T]$ and for all $\varepsilon > 0$,*

$$\sup_{\varphi \in \mathcal{S}^M_{+,\varepsilon}} \int_{I\times\mathbb{R}^d} |z|^2 \varphi(s,z)\nu(dz)ds \le \tau(a^2(\varepsilon) + |I|).$$

Proof. Fix $I \subset [0,T]$ measurable. Due to **Remark 1.1.5** $\nu(\mathbb{R}^d) < \infty$. *Young's inequality* (**Lemma D.7.1**) yields for every $\sigma \ge 1$

$$\begin{aligned} \int_I \int_{\mathbb{R}^d} |z|^2\varphi(s,z)\nu(dz)ds &\le \int_{I\times\mathbb{R}^d} e^\sigma \nu(dz)ds + \frac{1}{\sigma}\int_{I\times\mathbb{R}^d} \ell(|z|^2\varphi(s,z))\nu(dz)ds \\ &\le e^\sigma \nu(\mathbb{R}^d)|I| + \frac{1}{\sigma}\int_{I\times\mathbb{R}^d} \ell(|z|^2\varphi(s,z))\nu(dz)ds. \end{aligned} \tag{3.2.1}$$

We define

$$E := \Big\{(s,z) \in [0,T]\times\mathbb{R}^d \mid |z|^2\varphi(s,z) \ge 1\Big\}$$

and divide

$$\begin{aligned} &\int_{I\times\mathbb{R}^d} \ell(|z|^2\varphi(s,z))\nu(dz)ds \\ &= \int_{(I\times\mathbb{R}^d)\cap E} \ell(|z|^2\varphi(s,z))\nu(dz)ds + \int_{(I\times\mathbb{R}^d)\cap E^c} \ell(|z|^2\varphi(s,z))\nu(dz)ds. \end{aligned} \tag{3.2.2}$$

On E^c the condition $|z|^2\varphi(s,z) \le 1$ implies that $\ell(|z|^2\varphi(s,z)) \le 1$ and consequently

$$\int_{(I\times\mathbb{R}^d)\cap E^c} \ell(|z|^2\varphi(s,z))\nu(dz)ds \le \nu(\mathbb{R}^d)|I| < \infty.$$

On E we have $|z|^2\varphi(s,z) \ge 1$. *Young's inequality*, the non-decreasing behaviour and the convexity of ℓ on $[1,+\infty)$ yield

$$\begin{aligned} \ell(|z|^2\varphi(s,z)) &\le \ell\Big(\frac{1}{p}|z|^{2p} + \frac{1}{q}(\varphi(s,z))^q\Big) \\ &\le \frac{1}{p}\ell(|z|^{2p}) + \frac{1}{q}\ell((\varphi(s,z))^q), \end{aligned}$$

for any conjugate exponents $p, q \in [1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Consequently,

$$\begin{aligned} &\int_{(I\times\mathbb{R}^d)\cap E} \ell(|z|^2\varphi(s,z))\nu(dz)ds \\ &\le \frac{1}{p}\int_{(I\times\mathbb{R}^d)\cap E} \ell(|z|^{2p})\nu(dz)ds + \int_{(I\times\mathbb{R}^d)\cap E} \frac{1}{q}\ell(\varphi(s,z)^q)\nu(dz)ds. \end{aligned} \tag{3.2.3}$$

Due to *Fatou's lemma*,

$$\limsup_{q\to 1}\int_{I\times\mathbb{R}^d}\ell(\varphi(s,z)^q)\nu(dz)ds \le \int_{I\times\mathbb{R}^d}\limsup_{q\to 1}\ell(\varphi(s,z)^q)\nu(dz)ds$$
$$\le \int_{[0,T]\times\mathbb{R}^d}\ell(\varphi(s,z))\nu(dz)ds$$
$$\le Ma^2(\varepsilon).$$

Therefore, there exists $\delta > 0$ such that, for some $1+\delta > q_0 > 1$,

$$\frac{1}{q_0}\int_{[0,T]\times\mathbb{R}^d}\ell(g(s,z)^{q_0})\nu(dz)ds \le \frac{Ma^2(\varepsilon)}{q_0}.$$

Hence for $p_0 := \dfrac{q_0}{q_0-1}$ the corresponding convex conjugate we have

$$\begin{aligned}
&\int_{(I\times\mathbb{R}^d)\cap E}\ell(|z|^2\varphi(s,z))\nu(dz)ds \\
&\le \frac{1}{p_0}\int_{(I\times\mathbb{R}^d)\cap E}\ell(|z|^{2p_0})\nu(dz)ds + \frac{1}{q_0}\int_{(I\times\mathbb{R}^d)\cap E}\ell((g(s,z))^{q_0})\nu(dz)ds \\
&\le \int_{I\times\mathbb{R}^d}\ell(|z|^{2p_0})e^{-|z|^\alpha}dzds + \frac{Ma^2(\varepsilon)}{q_0}.
\end{aligned} \tag{3.2.4}$$

There exists $R_1 > 0$ such that $\ell(|z|^{2p_0}) \le |z|^{2p_0+1}$ in $\{|z| \ge R_1\}$ which implies

$$\begin{aligned}
&\int_{I\times\mathbb{R}^d}\ell(|z|^{2p_0})\nu(dz)ds \\
&\le \int_{I\times\{|z|\ge R_1\}}|z|^{2p_0+1}\nu(dz)ds + \int_{I\times\{|z|\le R_1\}}\ell(|z|^{2p_0})\nu(dz)ds.
\end{aligned} \tag{3.2.5}$$

Since ℓ is bounded in $|z| \le R_1$ and $\nu(\mathbb{R}^d) < \infty$, the second integral is bounded by $C_1|I|$, for some $C_1 > 0$.
Using first the genelized spherical change of coordinates in $\mathbb{R}^d$ and in the second line below the change of coordinates $y = x^\alpha$, we have

$$\begin{aligned}
\int_{I\times\mathbb{R}^d}|z|^{2p_0+1}e^{-|z|^\alpha}dzds &\le 2|I|\pi^{d-1}\int_0^\infty x^{2p_0+1}x^{d-1}e^{-x^\alpha}dx \\
&= 2|I|\pi^{d-1}\int_0^\infty y^{\frac{2p_0+d+1}{\alpha}-1}e^{-y}dy \\
&\le C_2|I|,
\end{aligned} \tag{3.2.6}$$

for $C_2 = 2\pi^{d-1}\Gamma\Big(\frac{2p_0+d+1}{\alpha}\Big) > 0$, where Γ is the Euler Γ- function, defined in (1.1.7).
Collecting (3.2.1), (3.2.2), (3.2.3), (3.2.4), (3.2.5) and (3.2.6) the result follows. $\square$

Lemma 3.2.2. *Let I be a measurable set of $[0,T]$ and $M>0$. Then there exist maps $\zeta,\rho:(0,\infty)\longrightarrow(0,\infty)$ such that $\zeta(u)\to 0$ as $u\to 0$, for all $\varepsilon,\beta\in(0,+\infty)$ we have*

$$\sup_{\psi\in\mathcal{S}_\varepsilon^M}\int_{I\times\mathbb{R}^d}|z\psi(z,s)|\mathbf{1}_{\{|\psi|\geq\frac{\beta}{a(\varepsilon)}\}}(s,z)\nu(dz)ds\leq\zeta(\beta)(1+|I|^{1/2}) \tag{3.2.7}$$

and

$$\sup_{\psi\in\mathcal{S}_\varepsilon^M}\int_{I\times\mathbb{R}^d}|z\psi(z,s)|\nu(dz)ds\leq\rho(\beta)|I|^{1/2}+\zeta(\beta)a(\varepsilon). \tag{3.2.8}$$

Proof. Let us fix arbitrarily $\psi\in\mathcal{S}_\varepsilon^M$ and $\beta>0$. Hence,

$$\begin{aligned}\int_{I\times\mathbb{R}^d}|z\psi(s,z)|\nu(dz)ds\leq&\int_{I\times\mathbb{R}^d}|z\psi(s,z)|\mathbf{1}_{\{|\psi|<\frac{\beta}{a(\varepsilon)}\}}(s,z)\nu(dz)ds\\&+\int_{I\times\mathbb{R}^d}|z\psi(s,z)|\mathbf{1}_{\{|\psi|\geq\frac{\beta}{a(\varepsilon)}\}}(s,z)\nu(dz)ds.\end{aligned} \tag{3.2.9}$$

The *Cauchy-Schwarz inequality* and **Lemma D.7.2- (c)** imply, for the constant $\kappa_2>0$ fixed there and for $c_\nu^2=\int_{\mathbb{R}^d}|z|^2\nu(dz)ds<\infty$, that

$$\begin{aligned}&\int_{I\times\mathbb{R}^d}|z\psi(s,z)|\mathbf{1}_{\{|\psi|<\frac{\beta}{a(\varepsilon)}\}}(s,z)\nu(dz)ds\\&\leq\Big(|I|\int_{\mathbb{R}^d}|z|^2\nu(dz)\int_{I\times\mathbb{R}^d}|\psi(s,z)|^2\mathbf{1}_{\{|\psi|<\frac{\beta}{a(\varepsilon)}\}}(s,z)\Big)^{1/2}\nu(dz)ds\\&\leq(c_\nu^2)^{1/2}(M\kappa_2(\beta))^{1/2}|I|^{1/2}.\end{aligned} \tag{3.2.10}$$

We recall that $\varphi=1+a(\varepsilon)\psi$ and note that $\varphi\in\mathcal{S}_{+,\varepsilon}^M$. In order to bound the second term in (3.2.9) we apply the *Cauchy-Schwarz inequality* and **Lemma D.7.2-(a)**. Hence

$$\begin{aligned}&\int_{I\times\mathbb{R}^d}|z\psi(s,z)|\mathbf{1}_{\{|\psi|\geq\frac{\beta}{a(\varepsilon)}\}}(s,z)\nu(dz)ds\\&\leq\Big(\int_{I\times\mathbb{R}^d}|z|^2|\psi(z,s)|\nu(dz)ds\int_{I\times\mathbb{R}^d}|\psi(z,s)|\mathbf{1}_{\{|\psi|\geq\frac{\beta}{a(\varepsilon)}\}}(s,z)\nu(dz)ds\Big)^{1/2}\\&\leq\Big(Ma(\varepsilon)\kappa_1(\beta)\int_{I\times\mathbb{R}^d}|z|^2|\psi(s,z)|\nu(dz)ds\Big)^{1/2}\\&\leq\Big(M\kappa_1(\beta)\int_{I\times\mathbb{R}^d}|z|^2|\varphi(z,s)-1|\nu(dz)ds\Big)^{1/2}\\&\leq(M\kappa_1(\beta)\Big(|I|c_\nu^2+\int_{\times I\times\mathbb{R}^d}|z|^2\varphi(s,z)\nu_T(dsdz)\Big)\Big)^{1/2}\\&\leq(M\kappa_1(\beta))^{1/2}(c_\nu^2|I|+\tau(a^2(\varepsilon)+|I|))^{1/2},\end{aligned} \tag{3.2.11}$$

where we obtained the last inequality in virtue of **Lemma 3.2.1** . Recall from **Lemma D.7.1-(b)** that $\kappa_1(\beta)\to 0$ as $\beta\to\infty$. The statement (3.2.7) follows immediately from the last inequality in (3.2.11) and (3.2.8) combining (3.2.10) and (3.2.11). □

Lemma 3.2.3. *For every $M > 0$ and $\delta > 0$ there exists a compact set $C_\delta \subset \mathbb{R}^d$ such that*

$$\sup_{\varepsilon>0} \sup_{\psi \in \mathcal{S}_\varepsilon^M} \int_{C_\delta^c \times [0,T]} |z\psi(s,z)|\nu(dz)ds < \delta.$$

Proof. Due to **Lemma 3.2.2** we have for all $\psi \in \mathcal{S}_\varepsilon^M$,

$$\int_0^T \int_{\mathbb{R}^d} |z\psi(s,z)|\mathbf{1}_{\{|\psi| \geq \frac{\beta}{a(\varepsilon)}\}}(s,z)\nu(dz)ds \leq \zeta(\beta)(1 + T^{1/2}),$$

where $\zeta(\beta) \to 0$ as $\beta \to \infty$. We choose $\beta_0 > 0$ large enough such that

$$\zeta(\beta_0)(1 + T^{1/2}) < \frac{\delta}{2}. \tag{3.2.12}$$

Next from **Lemma D.7.2- (c)** we derive for any compact $C \subset \mathbb{R}^d$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |z\psi(s,z)|\mathbf{1}_{\{|\psi| \leq \frac{\beta_0}{a(\varepsilon)}\}}(s,z)\nu(dz)ds \\
&\leq \Big(T \int_{C^c} |z|^2 \nu(dz) \int_0^T \int_{\mathbb{R}^d} \psi^2 \mathbf{1}_{\{|\psi| \leq \beta_0/a(\varepsilon)\}}(s,z)\Big)^{1/2} \\
&\leq \Big(MT\kappa_2(\beta_0) \int_{C^c} |z|^2 \nu(dz)\Big)^{1/2}.
\end{aligned}$$

Therefore, since

$$\int_{\mathbb{R}^d} |z|^2 \nu(dz) < \infty,$$

we find a compact C_δ such that

$$\Big(MT\kappa_2(\beta_0) \int_{C^c} |z|^2 \nu(dz)\Big)^{1/2} < \delta/2.$$

Combining (3.2.12) and the last estimate, this finishes the proof. □

Lemma 3.2.4. *For every $M > 0$ and $\varepsilon > 0$ we have*

$$\sup_{\psi \in \mathcal{S}_\varepsilon^M} \int_0^T \int_{\mathbb{R}^d} |z||\psi(s,z)|\mathbf{1}_{\{|\psi| > \beta/a(\varepsilon)\}}(s,z)\nu(dz)ds \to 0 \quad \text{as } \beta \to \infty.$$

Proof. In view of **Lemma 3.2.3** it suffices to show that for any compact $C \subset \mathbb{R}^d$

$$\sup_{\psi \in \mathcal{S}_\varepsilon^M} \int_0^T \int_C |z\psi(s,z)|\mathbf{1}_{\{|\psi| > \beta/a(\varepsilon)\}}(s,z)\nu(dz)ds \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.2.13}$$

For given $\Theta > 0$ and every $z \in \mathbb{R}^d$ we write

$$z = z\mathbf{1}_{\{|z| \leq \Theta\}} + z\mathbf{1}_{\{|z| > \Theta\}}.$$

From **Lemma D.7.2- (a)** we conclude, for any $\psi \in \mathcal{S}_\varepsilon^M$,

$$\begin{aligned}
&\int_0^T \int_C z\mathbf{1}_{\{|z|\le\Theta\}}(s,z)|\psi(s,z)|\mathbf{1}_{\{|\psi|>\beta/a(\varepsilon)\}}(s,z)\nu(dz)ds \\
&\le \int_{[0,T]\times C} |\psi(s,z)|\mathbf{1}_{\{|\psi|>\beta/a(\varepsilon)\}}(s,z)\nu(dz)ds \\
&\le \Theta M\kappa_1(\beta)a(\varepsilon). \qquad (3.2.14)
\end{aligned}$$

From **Lemma D.7.1** we have $\kappa_1(\beta) \to 0$ as $\beta \to \infty$ and therefore the above estimate converges to zero as $\beta \to \infty$.

The same calculation as in (3.2.11) but with $\mathbb{R}^d$ replaced by C and z by $z\mathbf{1}_{\{z>\Theta\}}$ yields for $\varphi = 1 + a(\varepsilon)\psi$ and for some $\tau > 0$ given by **Lemma 3.2.1**

$$\begin{aligned}
&\int_0^T \int_C |z|\mathbf{1}_{\{|z|>\Theta\}}(z)|\psi(s,z)|\mathbf{1}_{\{|\psi|>\beta/a(\varepsilon)\}}(s,z)\nu(dz)ds \\
&\le \Big(M\kappa_1(\beta)\Big(T\int_C |z|^2\mathbf{1}_{\{|z|>\Theta\}}(z)\nu(dz) + \int_0^T\int_C |z|^2\mathbf{1}_{\{|z|>\Theta\}}(z)\varphi(s,z)d\nu(dz)ds\Big)\Big)^{1/2} \\
&\le M\kappa_1(\beta)\Big(Tc_\nu^2 + \tau(a^2(\varepsilon)+T)\Big)^{1/2}. \qquad (3.2.15)
\end{aligned}$$

Sending $\beta \to \infty$ the right hand-side of the above estimate converges to zero. □

Lemma 3.2.5. *Fix $M > 0$. Let $(\psi_\varepsilon)_{\varepsilon>0}$ such that for every $\varepsilon > 0$ $\psi_\varepsilon \in \mathcal{U}_\varepsilon^M$. We assume that, for some $\beta \in (0,1]$, $\psi_\varepsilon \mathbf{1}_{\{|\psi_\varepsilon|\le\frac{\beta}{a(\varepsilon)}\}}$ converges in law in $B^2(\sqrt{M\kappa_2(1)})$ to ψ. Then we have*

$$\int_0^t\int_{\mathbb{R}^d} z\psi_\varepsilon(s,z)\nu(dz)ds \to \int_0^t\int_{\mathbb{R}^d} z\psi(s,z)\nu(dz)ds \quad \textit{for all } t\in[0,T],$$

in distribution.

Proof. From **Lemma 3.2.4** we have that

$$\int_0^T\int_{\mathbb{R}^d} |z\psi_\varepsilon(s,z)|\mathbf{1}_{\{|\psi_\varepsilon|>\beta/a(\varepsilon)\}}(s,z)\nu(dz)ds \to 0 \quad \text{as } \varepsilon \to 0.$$

Also, since $z\mathbf{1}_{[0,t]} \in L^2(\nu_T)$ for all $t \in [0,T]$ and $z \in \mathbb{R}^d$, due to the weak convergence in $B^2(\sqrt{M\kappa_2(1)})$ of

$$\psi_\varepsilon\mathbf{1}_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}} \to \psi,$$

we have

$$\int_0^t\int_{\mathbb{R}^d} z\psi_\varepsilon(s,z)\mathbf{1}_{\{|\psi_\varepsilon|\le\beta/a(\varepsilon)\}}\nu(dz)ds \to \int_0^t\int_{\mathbb{R}^d} z\psi(s,z)\nu(dz)ds.$$

The result follows combining the last two statements. □

Remark 3.2.1. For fixed $M > 0$ and $\varepsilon > 0$ and $\varphi_\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$ we denote by $N^{\frac{1}{\varepsilon}\varphi_\varepsilon}$ the controlled random measure

$$N^{\frac{1}{\varepsilon}\varphi_\varepsilon}([0,t]\times U) := \int_0^t \int_U \int_0^\infty \mathbf{1}_{[0,\frac{1}{\varepsilon}\varphi_\varepsilon]}(r)\bar{N}(ds,dz,dr) \quad \text{for all } t \in [0,T], U \in \mathcal{B}(\mathbb{R}^d).$$

For every $\varepsilon > 0$, we define $\tilde{\varphi}_\varepsilon = \frac{1}{\varphi_\varepsilon}$. We note that this implies that $\tilde{\varphi}_\varepsilon$ is bounded below and above on a certain compact $K_j \in (K_n)_{n\in\mathbb{N}}$ and $\tilde{\varphi}_\varepsilon = 1$ outside of K_j.
Girsanov's theorem (**Theorem B.3.2**) implies that the stochastic exponential, defined for $t \in [0,T]$ as

$$\begin{aligned}&\mathcal{E}(\tilde{\varphi}_\varepsilon)(t)\\ &:= \exp\Big(\int_0^t\int_{\mathbb{R}^d}\int_0^{\frac{1}{\varepsilon}} \ln\tilde{\varphi}_\varepsilon(s,z)\bar{N}(ds,dz,dr) + \int_0^t\int_{\mathbb{R}^d}\int_0^{\frac{1}{\varepsilon}}(-\tilde{\varphi}_\varepsilon(s,z)+1)\nu(dz)drds\Big)\end{aligned}$$

is a $(\bar{\mathcal{F}}_t)_{0\le t\le T}$ - martingale and the measure defined as

$$\mathbb{Q}^\varepsilon_T(G) := \int_G \mathcal{E}(\tilde{\varphi}_\varepsilon)(T)d\bar{\mathbb{P}}(\bar{m}) \quad \text{for all } G \in \mathcal{B}(\bar{\mathbb{M}})$$

is a probability measure on $(\bar{\mathbb{M}},\mathcal{B}(\bar{\mathbb{M}}))$. $\bar{\mathbb{P}}$ and $\mathbb{Q}^\varepsilon_T$ are mutually absolutely continuous and the controlled Poisson random measure $\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}$ has the same law under $\mathbb{Q}^\varepsilon_T$ as $\varepsilon N^{\frac{1}{\varepsilon}}$ under $\bar{\mathbb{P}}$. We do not stress the dependence of the integral with respect to $\mathbb{Q}^\varepsilon_T$.

We recall the process $\tilde{X}^\varepsilon$ is the unique strong solution of the following controlled SDE, for all $t \in [0,T]$,

$$\tilde{X}^\varepsilon_t = x - \int_0^t \nabla U(\tilde{X}^\varepsilon_s)ds + \int_0^t\int_{\mathbb{R}^d} z(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dsdz) - \nu(dz)ds). \tag{3.2.16}$$

For sake of readibility, we omit the dependence of $(X^{\varepsilon,x}_t)_{0\le t\le T}$ on the initial condition $x \in \mathbb{R}^d$.
We recall **Proposition 2.2.1**. There exists $\varepsilon_0 > 0$ such that

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\Big[\sup_{0\le s\le T}|\tilde{X}^\varepsilon_s|^2\Big] < \infty. \tag{3.2.17}$$

For every $\varepsilon > 0$ and for any initial value $x \in \mathbb{R}^d$ we define now

$$\tilde{Y}^{\varepsilon,x} := \frac{1}{a(\varepsilon)}(\tilde{X}^{\varepsilon,x} - X^{0,X}).$$

Proposition 3.2.1 (**Uniform bound for the controlled averaged process** $(\tilde{Y}^\varepsilon_t)_{0\le t\le T}$). *There exists some $\varepsilon_0 > 0$ such that, for any initial value $x \in \mathbb{R}^d$, we have*

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\Big[\sup_{0\le s\le T}|\tilde{Y}^\varepsilon_s|^2\Big] < \infty. \tag{3.2.18}$$

Proof. For every $\varepsilon > 0$ and $M > 0$ we fix arbitrarily $\varphi_\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$. Let $\psi_\varepsilon = \frac{\varphi_\varepsilon - 1}{a(\varepsilon)}$. Since

$$\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz) = N^{\frac{1}{\varepsilon}\varphi}(ds,dz) - \frac{1}{\varepsilon}\varphi_\varepsilon(s,z)ds\nu(dz),$$

we obtain, for every $t \in [0,T]$,

$$\begin{aligned}\tilde{X}^\varepsilon_t - X^0_t = &\int_0^t (-\nabla U(\tilde{X}^\varepsilon_s) + \nabla U(X^0_s))ds + \varepsilon \int_0^t \int_{\mathbb{R}^d} z\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz)\\ &+ \int_0^t \int_{\mathbb{R}^d} z(\varphi_\varepsilon(s,z) - 1)\nu(dz)ds.\end{aligned}$$

We write, for every $\varepsilon > 0$,

$$\tilde{Y}^\varepsilon = A^\varepsilon + M^\varepsilon + B^\varepsilon,$$

where, for all $t \in [0,T]$,

$$\begin{aligned}A^\varepsilon_t &= \frac{1}{a(\varepsilon)} \int_{[0,t]} (-\nabla U(\tilde{X}^\varepsilon_s) + \nabla U(X^0_s))ds,\\ M^\varepsilon_t &= \frac{\varepsilon}{a(\varepsilon)} \int_0^t \int_{\mathbb{R}^d} z\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz) \text{ and}\\ B^\varepsilon_t &= \int_0^t \int_{\mathbb{R}^d} z\psi^\varepsilon(s,z)\nu(dz)ds.\end{aligned}$$

Due to **Condition 1.1.1** and (2.2.1), there exist $\varepsilon_0 > 0$ and $R > 0$ such that for $0 < \varepsilon < \varepsilon_0$

$$\bar{\mathbb{E}}\Big[\sup_{0\le s\le t} |\tilde{X}^\varepsilon_s|^2\Big] \le R.$$

Therefore for some $C = C_R > 0$

$$\bar{\mathbb{E}}\Big[\sup_{0\le r\le t} |A^\varepsilon_r|^2\Big] \le C\int_0^t \bar{\mathbb{E}}\Big[\sup_{0\le r\le s} |\tilde{Y}^\varepsilon_r|^2\Big]ds. \tag{3.2.19}$$

Noting that M^ε is a martingale and due to **Lemma 3.2.1**, there exists some constant $\tau > 0$ such that

$$\begin{aligned}\bar{\mathbb{E}}\Big[\sup_{0\le r\le T} |M^\varepsilon_r|^2\Big] &\le \Big(\frac{\varepsilon}{a(\varepsilon)}\Big)^2 \bar{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{R}^d} |z|^2 \frac{1}{\varepsilon}\varphi(s,z)\nu(dz)ds\Big]\\ &\le \frac{\varepsilon}{a^2(\varepsilon)}(\tau(a^2(\varepsilon) + T)).\end{aligned} \tag{3.2.20}$$

Lemma 3.2.2 implies, for some $\rho(\beta), \zeta(\beta) > 0$, that

$$\begin{aligned}\sup_{0\leq r\leq t} |B_r|^2 &\leq \Big(\int_0^t \int_{\mathbb{R}^d} |z|\psi^\varepsilon(s,z)\nu(dz)ds\Big)^2 \\ &\leq (\rho(\beta)T + \zeta(\beta)a(\varepsilon))^2. \end{aligned} \tag{3.2.21}$$

Combining (3.2.19), (3.2.20) and (3.2.21) and applying *Gronwall's inequality* yield for $\varepsilon < \varepsilon_0$

$$\begin{aligned}\bar{\mathbb{E}}\Big[\sup_{0\leq t\leq T} |\tilde{Y}_t^\varepsilon|^2\Big] &\leq \Big(\frac{\varepsilon}{a^2(\varepsilon)}(\tau(a^2(\varepsilon)+T)) + (\rho(\beta)T + \zeta(\beta)a(\varepsilon))^2\Big)e^{CT} \\ &< \infty. \end{aligned} \tag{3.2.22}$$

□

Proof of Theorem 1.3.2

Proof of Theorem 1.3.2. For every $\varepsilon > 0$ there exists a measurable map

$$\mathcal{G}^\varepsilon : \mathbb{M} \longrightarrow \mathbb{D}([0,T], \mathbb{R}^d),$$

such that

$$Y^\varepsilon =: \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}}).$$

We define

$$\begin{aligned}\mathcal{G}^0 : L^2(\nu_T) &\longrightarrow C([0,T], \mathbb{R}^d) \\ \mathcal{G}^0(\psi) &:= \eta,\end{aligned}$$

where $\eta \in C([0,T,\mathbb{R}^d])$ is the solution of

$$\eta(T) = -\int_0^t \nabla^2 U(X^0(s))\eta_s ds + \int_0^t \int_{\mathbb{R}^d} \psi(s,z)z\nu(dz)ds, \quad t \in [0,T].$$

In order to prove that $(Y^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle with speed ε^α and with good rate function

$$\begin{aligned}\tilde{\mathbb{I}}_1 : \mathbb{D}([0,T], \mathbb{R}^d) &\longrightarrow [0,\infty] \\ \tilde{\mathbb{I}}_1(\eta) &:= \sup_{\psi\in\mathbb{T}_\eta} \frac{1}{2}||\psi||^2_{L^2(\nu_T)},\end{aligned}$$

we check **Condition 3.1.1**.

(i) Given $M > 0$, for every $\varepsilon > 0$ let $g_\varepsilon, g \in B^2(M)$ and $g_\varepsilon \to g$ in the weak-topology of $L^2(\nu_T)$. We prove that $\mathcal{G}^0(g)$ is a weak-$L^2(\nu_T)$ limit point of $\mathcal{G}^0(g_\varepsilon)$, when $\varepsilon \to 0$.

We set $\mathcal{G}^0(g_\varepsilon) = \eta_\varepsilon$, where

$$\eta_\varepsilon(t) = -\int_0^t \nabla U(\eta_\varepsilon(s))ds + \int_0^t \int_{\mathbb{R}^d} z g_\varepsilon(s,z)\nu(dz)ds, \quad t \in [0,T]. \tag{3.2.23}$$

and $\mathcal{G}^0(g) = \eta$ with

$$\eta(t) = -\int_0^t \nabla U(\eta(s))ds + \int_0^t \int_{\mathbb{R}^d} z g(s,z)\nu(dz)ds, \quad t \in [0,T]. \tag{3.2.24}$$

Since $g_\varepsilon \rightharpoonup g$ in the weak topology of $L^2(\nu_T)$ we have

$$\langle g_\varepsilon, z\rangle_{L^2(\nu_T)} \to \langle g, z\rangle_{L^2(\nu_T)}, \qquad \text{as } \varepsilon \to 0.$$

where $\langle .,. \rangle_{L^2(\nu_T)}$ is the inner product in $L^2(\nu_T)$. Therefore,

$$\int_0^t \int_{\mathbb{R}^d} z g_\varepsilon(s,z)\nu(dz)ds \to \int_0^t \int_{\mathbb{R}^d} z g(s,z)\nu(dz)ds \quad \text{ for all } t \geq 0,$$

as $\varepsilon \to 0$. Since $g \in L^2(\nu_T)$, there exists some $R > 0$ such that

$$\sup_{\varepsilon>0} \sup_{0\leq t\leq T} |\eta_\varepsilon(t)| < R.$$

Due to **Condition 1.1.1** there is $C = C_R > 0$ be such that

$$|\nabla U(x)| \leq C \quad \text{for all } x \in B_R(0).$$

The *Cauchy-Schwarz inequality* implies that for every $0 \leq s \leq t \leq T$ and $\varepsilon > 0$,

$$\begin{aligned}
&|\eta_\varepsilon(t) - \eta_\varepsilon(s)| \\
&\leq \int_s^t |\nabla U(\eta_\varepsilon(r))|dr + \int_s^t \int_{\mathbb{R}^d} |z||g_\varepsilon(s,z)|\nu(dz)ds \\
&\leq \int_s^t |\nabla U(\eta_\varepsilon(r))|dr + \Big(\int_s^t \int_{\mathbb{R}^d} |g_\varepsilon(s,z)|^2\nu(dz)ds\Big)^{1/2} \Big(\int_s^t \int_{\mathbb{R}^d} |z|^2\nu(dz)ds\Big)^{1/2} \\
&\leq C|t-s| + M\sqrt{c_\nu^2|t-s|},
\end{aligned}$$

where $c_\nu^2 := \displaystyle\int_{\mathbb{R}^d} |z|^2\nu(dz) < \infty$.

The preceding estimate shows that $(\eta_\varepsilon)_{\varepsilon>0}$ is a equicontinuous family of functions of $C([0,T],\mathbb{R}^d)$. Since $(\eta_\varepsilon)_{\varepsilon>0}$ is uniformly bounded, due to *Arzela-Ascoli's theorem* (**Proposition A.1.4**) let $\tilde{\eta}$ be a limit point of $(\eta_\varepsilon)_{\varepsilon>0}$ in $C([0,T],\mathbb{R}^d)$ for the uniform topology. The weak convergence in $L^2(\nu_T)$ $g_\varepsilon \rightharpoonup g$ as $\varepsilon \to 0$ imply that, for all $t \in [0,T]$

$$\int_0^t \int_{\mathbb{R}^d} z g_\varepsilon(s,z)\nu(dz)ds \to \int_0^t \int_{\mathbb{R}^d} z g(s,z)\nu(dz)ds, \quad \text{as } \varepsilon \to 0.$$

The continuity of ∇U and dominated convergence imply that we can perform the pointwise limit in the first term of the right hand-side (3.2.23). The weak convergence in $L^2(\nu_T)$ $g_\varepsilon \rightharpoonup g$ as $\varepsilon \to 0$ permits to do the limit in the second integral in the right-hand side of (3.2.23). Finally, the pointwise convergence of η_ε for the limit point η implies that

$$\tilde{\eta}(t) = -\int_0^t \nabla U(\tilde{\eta}(s))ds + \int_0^t \int_{\mathbb{R}^d} zg(s,z)\nu(dz)ds, \quad t \in [0,T].$$

Since (3.2.24) has a unique continuous solution we conclude that $\tilde{\eta} = \eta$, which finishes the proof that $\mathcal{G}^0(g)$ is a a limit point of $(\mathcal{G}^\varepsilon(g_\varepsilon))_{\varepsilon>0}$.

(ii) Given $M > 0$, let $(\varphi_\varepsilon)_{\varepsilon>0} \subset \mathcal{U}^M_{+,\varepsilon}$. For some $\beta \in (0,1]$ we assume $\psi_\varepsilon \mathbf{1}_{\{|\psi_\varepsilon| \leq \beta/a(\varepsilon)\}} \Rightarrow \psi$ in $B^2(\sqrt{Mk_2(1)})$, where $\psi_\varepsilon = \frac{\varphi_\varepsilon - 1}{a(\varepsilon)}$. We prove that $\mathcal{G}^0(\psi)$ is a weak limit point of $\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon})$, as $\varepsilon \to 0$.
In order to prove that the family $(\tilde{Y}^\varepsilon)_{\varepsilon>0}$ has a limit point in $\mathbb{D}([0,T],\mathbb{R}^d)$, as $\varepsilon \to 0$ we use *Prokhorov's theorem* (**Proposition C.1.5**). In order to prove that $(\tilde{Y}^\varepsilon)_{\varepsilon>0}$ is a tight family in the Skorokhod space we use the tightness criteria given in **Proposition C.2.3**.

Using the notation of **Proposition 3.2.1**, we write, for every $\varepsilon > 0$,

$$\tilde{Y}^\varepsilon = A^\varepsilon + M^\varepsilon + B^\varepsilon,$$

where, for all $t \in [0,T]$,

$$\begin{aligned}
A^\varepsilon_t &= \frac{1}{a(\varepsilon)} \int_{[0,t\times]\mathbb{R}^d} (-\nabla U(\tilde{X}^\varepsilon_s) + \nabla U(X^0_s))ds,\\
M^\varepsilon_t &= \frac{\varepsilon}{a(\varepsilon)} \int_{[0,t]\times\mathbb{R}^d} z\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds,dz) \text{ and}\\
B^\varepsilon_t &= \int_{[0,t]\times\mathbb{R}^d} \psi^\varepsilon(s,z)z\nu(dz)ds.
\end{aligned}$$

Let $C > 0$ be given in (3.2.19). *Chebyshev's inequality* yields, for every $\tau > 0$ and for

$$\delta = \delta_\tau < \frac{\tau^2}{C\bar{\mathbb{E}}\Big[\sup\limits_{0\leq t\leq T} |\tilde{Y}^\varepsilon_t|^2\Big]},$$

$$\begin{aligned}
\bar{\mathbb{P}}\Big(\sup_{0\leq t-s\leq\delta} |A^\varepsilon_t - A^\varepsilon_s| > \tau\Big) &\leq \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T} |\tilde{Y}^\varepsilon_t|^2 > \frac{\tau}{\delta C}\Big)\\
&\leq \frac{\delta C}{\tau}\bar{\mathbb{E}}\Big[\sup_{0\leq t\leq T} |\tilde{Y}^\varepsilon_t|^2\Big]\\
&\leq \tau.
\end{aligned}$$

As in the proof of **Theorem 1.2.1**, we conclude that $(A^\varepsilon)_{\varepsilon>0}$ is C-tight (see **Definition C.2.2**).
In order to conclude the tightness of $(M^\varepsilon)_{\varepsilon>0}$ it is enough to observe, by (3.2.20), that

$$\lim_{\varepsilon\to 0}\bar{\mathbb{E}}\Big[\sup_{0\le r\le T}|M_r^\varepsilon|^2\Big]=0.$$

Due to (3.2.21) and **Lemma 3.2.2**, we have, for every $r\in[0,T]$ and $\delta>0$,

$$|B_{r+\delta}^\varepsilon-B_r^\varepsilon|\le\rho(\beta)\delta+\zeta(\beta)a^2(\varepsilon)\to 0\quad\text{ as }\beta\to\infty,\varepsilon\to 0.$$

This implies that, for every $\tau>0$, there exists $\delta_\tau>0$ such that

$$\sup_{\varepsilon>0}\mathbb{P}\Big(\sup_{0<|t-s|<\delta}|B_t^\varepsilon-B_s^\varepsilon|>\tau\Big)<\tau.$$

Hence, observing that $K_\tau\subset C([0,T],\mathbb{R}^d)$ defined in (2.2.22) is relatively compact, we have

$$\bar{\mathbb{P}}(B^\varepsilon\notin K_\tau)\le\tau,$$

which shows that $(B^\varepsilon)_{\varepsilon>0}$ is tight.

Therefore, due to **Proposition C.2.3**, we conclude the tightness of the family $(\tilde{Y}^\varepsilon)_{\varepsilon>0}$ in $\mathbb{D}([0,T],\mathbb{R}^d)$.

We note the trivial identity, for every $\varepsilon>0$ and $s\in[0,T]$,

$$\tilde{X}_s^\varepsilon=X_s^0+a(\varepsilon)\tilde{Y}_s^\varepsilon,$$

where $(\tilde{X}_s^\varepsilon)_{s\in[0,T]}$ solves the controlled SDE (2.2.15). Using Taylor's theorem we have for $s\in[0,T]$,

$$-\nabla U(\tilde{X}_s^\varepsilon)+\nabla U(X_s^0)=a(\varepsilon)(-\nabla^2U(X_s^0))\tilde{Y}_s^\varepsilon+\mathcal{R}_s^\varepsilon,$$

for some $(\mathcal{R}_s^\varepsilon)_{s\in[0,T]}$, C_R a Lipschitz constant of ∇^2U in the ball of radius R, with $R>0$ big enough such that

$$\sup_{0\le t\le T}|X_t^0|^2\vee\bar{\mathbb{E}}\Big[\sup_{0\le s\le T}|\tilde{X}_s^\varepsilon|^2\Big]\le R$$

and

$$\bar{\mathbb{E}}\Big[\sup_{0\le r\le t}|\mathcal{R}_r^\varepsilon|\Big]\le C_Ra^2(\varepsilon)\bar{\mathbb{E}}\Big[\sup_{0\le r\le t}|\tilde{Y}_r^\varepsilon|^2\Big].$$

In view of (3.2.22), we have the convergence

$$\bar{\mathbb{E}}\Big[\sup_{0\le t\le T}|\mathcal{R}_t^\varepsilon|\Big]\to 0\quad\text{as }\varepsilon\to 0.\tag{3.2.25}$$

It follows that

$$A_t^\varepsilon = -\int_0^t \nabla^2 U(X_s^0)\tilde{Y}_s^\varepsilon ds + \int_0^t |\mathcal{R}_s^\varepsilon| ds.$$

In conclusion, for every $s \in [0,T]$,

$$\tilde{Y}_t^\varepsilon = -\int_0^t \nabla^2 U(X_s^0)\tilde{Y}_s^\varepsilon ds + \int_0^t |\mathcal{R}_s^\varepsilon| ds + M_t^\varepsilon + \int_0^t \int_{\mathbb{R}^d} \psi_\varepsilon(s,z) z\nu(dz)ds. \tag{3.2.26}$$

Since $(\tilde{Y}^\varepsilon)_{\varepsilon>0}$ is tight in $\mathbb{D}([0,T],\mathbb{R}^d)$, due to **Proposition C.1.6**, let $(\tilde{Y}_t)_{t\in[0,T]}$ be a limit point in law of the family $(\tilde{Y}^\varepsilon)_{\varepsilon>0}$. Due to **Proposition C.1.6**, $\tilde{Y}^\varepsilon \to \tilde{Y}$ in $\mathbb{D}([0,T],\mathbb{R}^d)$, $\bar{\mathbb{P}}$-a.s. as $\varepsilon \to 0$.
Due to **Lemma 3.2.5** we have for all $t \in [0,T]$,

$$\int_0^t \int_{\mathbb{R}^d} \psi_\varepsilon(s,z) z\nu(dz)ds \to \int_0^t \int_{\mathbb{R}^d} \psi(s,z) z\nu(dz)ds,$$

as $\varepsilon \to 0$ in distribution.
Since

$$\lim_{\varepsilon\to 0} \bar{\mathbb{E}}\Big[\int_0^T |\mathcal{R}_s^\varepsilon| ds\Big] = 0$$
$$\lim_{\varepsilon\to 0} \bar{\mathbb{E}}\Big[\sup_{0\le s\le T} |M_s^\varepsilon|^2\Big] = 0$$

the continuity of ∇U and the observation before implies that, for all $t \in [0,T]$ and $\bar{\mathbb{P}}$-a.s.

$$\tilde{Y}_t = -\int_0^t \nabla^2 U(\tilde{X}_s^0)\tilde{Y}_s ds + \int_0^t \int_{\mathbb{R}^d} \psi(s,z) z\nu(dz)ds.$$

Since the ODE above has a unique continuous solution we conclude that $\tilde{Y}_t = \mathcal{G}^0(\psi)$, $\bar{\mathbb{P}}$-a.s. This finishes the proof that $\mathcal{G}^0(\psi)$ is a limit point in law of $(\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}))_{\varepsilon>0}$. □

Proof of Theorem 1.3.1

Proof. Given $x \in \mathbb{R}^d$, we use **Condition 3.1.1** to prove a moderate deviations principle for $(X_t^{\varepsilon,x})_{t\in[0,T]}$, solution of (1.1.4) when ν is given bye (1.1.5) for some $\alpha \in (0,1)$, in the Skorokhod space $\mathbb{D}([0,T],\mathbb{R}^d)$, with speed ε^α and good rate function

$$\tilde{\mathbb{I}}_0 : \mathbb{D}([0,T],\mathbb{R}^d) \longrightarrow [0,\infty],$$

$$\tilde{\mathbb{I}}_0(\eta) = \begin{cases} 0, & \text{if } \eta = X^0, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 1.1.1 implies, for every $\varepsilon > 0$ that there exists a measurable map

$$\mathcal{G}^\varepsilon : \mathbb{M} \longrightarrow \mathbb{D}([0,T], \mathbb{R}^d)$$

defined by

$$\mathcal{G}^{\varepsilon,x}(\varepsilon N^{\frac{1}{\varepsilon}}) := X^{\varepsilon,x}.$$

For sake of simplicity, we omit the dependence on the initial condition $x \in \mathbb{R}^d$. The map

$$\mathcal{G}^0 : L^2(\nu_T) \longrightarrow C([0,T], \mathbb{R}^d)$$

is defined by

$$\mathcal{G}^0(\psi) := X^0,$$

where $X^0 \in C([0,T], \mathbb{R}^d)$ is the unique solution of

$$X_s^0 = x - \int_0^t \nabla U(X_s^0) ds, t \in [0,T].$$

1. We prove that, given $M > 0$ and for every $\varepsilon > 0$ $g_\varepsilon, g \in B^2(M)$ such that $g_\varepsilon \to g$ weakly as $\varepsilon \to 0$, we have $\mathcal{G}^0(g)$ is a limit point of $(\mathcal{G}^0(g_\varepsilon))_{\varepsilon>0}$. Due to the definition of $\mathcal{G}^0$, the result is clear.

2. Given $M > 0$ and $(\varphi_\varepsilon)_{\varepsilon>0} \subset \mathcal{U}^M_{+,\varepsilon}$. Given $\beta \in (0,1]$ we assume the convergence in law $\psi_\varepsilon \mathbf{1}_{\{|\psi_\varepsilon| \le \beta/a(\varepsilon)\}} \Rightarrow \psi$ in $B_2(\sqrt{Mk_2(1)})$, where $\psi_\varepsilon = \frac{\varphi_\varepsilon - 1}{a(\varepsilon)}$. We prove that

   $$\mathcal{G}^0(\psi) \text{ is a limit point in law of } (\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}))_{\varepsilon>0} \text{ as } \varepsilon \to 0.$$

   As it was pointed in the proof of **Theorem 1.2.1**, for every $\varepsilon > 0$, $\tilde{X}^\varepsilon =: \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon})$ is the solution of the controlled SDE, for all $t \in [0,T]$,

   $$\tilde{X}_t^\varepsilon = x - \int_0^t \nabla U(\tilde{X}_s^\varepsilon) ds + \varepsilon \int_0^t \int_{\mathbb{R}^d} z \tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(ds, dz) + \int_0^t \int_{\mathbb{R}^d} z(\varphi_\varepsilon(s,z) - 1)\nu(dz) ds. \tag{3.2.27}$$

As it was argueed in the proof of **Theorem 1.2.1**, it is proved that $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ is a tight family in $\mathbb{D}([0,T], \mathbb{R}^d)$. For every $\varepsilon > 0$, using *Prokhorov's theorem* (**Proposition C.1.5**) and *Skorokhod's representation theorem* (**Proposition C.1.7**), let $\bar{X}^\varepsilon, \bar{X} \in C([0,T], \mathbb{R}^d)$ such that, for every $\varepsilon > 0$, $\bar{X}^\varepsilon = \tilde{X}^\varepsilon$ in law and $\bar{X}^\varepsilon \to \bar{X}$, as $\varepsilon \to 0$, $\bar{\mathbb{P}}$-a.s. If we denote, for every $t \in [0,T]$ the martingale

$$M_t^\varepsilon = \int_0^t \int_{\mathbb{R}^d} z \varepsilon \tilde{N}^{\frac{1}{\varepsilon}\varphi^\varepsilon}(ds, dz),$$

the estimate (2.2.23) implies that $\bar{\mathbb{E}}[[M^\varepsilon]_T] \to 0$, as $\varepsilon \to 0$. **Lemma 3.2.1** implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}^d} z(\varphi_\varepsilon(s,z) - 1)\nu(dz)ds \\
&= \lim_{\varepsilon\to 0} a(\varepsilon) \int_0^t \int_{\mathbb{R}^d} z\psi^\varepsilon(s,z)\nu(dz)ds \\
&= 0.
\end{aligned}$$

Due to the continuity of $-\nabla U$ and the considerations above, we can perform the pointwise limit in (3.2.27) as $\varepsilon \to 0$, $\bar{\mathbb{P}}$-a.s. and conclude that

$$\bar{X}_t = x - \int_0^t \nabla U(\bar{X}_s)ds, \quad s \in [0,T],$$

which proves that $\bar{X} = X^0$, $\bar{\mathbb{P}}$-a.s. This finishes the proof. $\square$

3.3 The asymptotic first exit time

3.3.1 Continuity properties of the cost functional

The goal of this section is to prove **Theorem 1.3.3**.

Proposition 3.3.1. *There exist $M > 0$, $\rho_0 > 0$ and $T :]0, \rho_0[\longrightarrow \mathbb{R}^+$ with $\lim_{\rho_0 \to 0} T(\rho_0) = 0$ satisfying the following:*
for all $\rho \in]0, \rho_0[, x_0, y_0 \in \mathbb{R}^d$ there exists $\eta \in C([0, T(\rho)], \mathbb{R}^d)$ and $\varphi \in L^2(\nu_T)$, such that $\Phi(T(\rho)) = y_0$, solution of

$$\eta(s) = x_0 + \int_0^s -\nabla U^2(X_r^0)\eta(r)\mathrm{d}r + \int_0^s \int_{\mathbb{R}^d} z\psi(r,z)\nu(\mathrm{d}z)\mathrm{d}r \quad 0 \leq s \leq T(\rho),$$

where $X_s^0 = x - \int_0^s \nabla U(X_r^0)dr, \quad s \geq 0.$

In particular for $\bar{V}_1$ defined in (1.3.4) we have $\bar{V}_1 < \infty$.

Proof. Let us fix $\rho > 0$ and $x, y \in \mathbb{R}^d$. We consider the straight line that links x and y,

$$\Phi(t) = x + t\frac{y-x}{\rho}, \quad t \in [0, \rho].$$

For every $t \in [0, \rho]$ we write

$$P_{x,y}(t) = \frac{y-x}{\rho} + \nabla^2 U(X_s^0)\Big(x + s\frac{y-x}{\rho}\Big).$$

We define

$$\begin{aligned} &\psi : [0, \rho] \times \mathbb{R}^d \longrightarrow [0, \infty) \\ &\psi(t,z) = \frac{e^{|z|^\alpha}}{\lambda^d(B_1(P_{x,y}(t)))}\mathbf{1}_{B_1(P_{x,y}(t))}(z). \end{aligned}$$

It is immediate, by definition, that ψ is a bounded function, which implies that $\psi \in L^2(\nu_\rho)$. Furthermore, η and ψ solve, for every $s \in [0, \rho]$,

$$\eta(s) = x - \int_0^s \nabla^2 U(X_r^0)\eta_r dr + \int_0^s \int_{\mathbb{R}^d} z\psi(s,z)\nu(dz)ds.$$

There exists $C > 0$ such that $|\psi(t,z)| \leq C$ for every $(t,z) \in [0, \rho] \times \mathbb{R}^d$. Hence,

$$\bar{V}_1 \leq V_1(x,y,\rho) \leq \frac{1}{2}C^2\nu(\mathbb{R}^d)\rho.$$

Writing $T(\rho) = \rho$ the conclusion follows. □

It is immediate the corollary.

Corollary 3.3.1. *For any $\delta > 0$, there exists $\rho > 0$ such that:*

i) $\sup_{|x|,|y|\leq\rho} \inf_{t\in[0,1]} V_1(x,y,t) < \delta$

ii) $\sup_{\{x,y:\inf_{z\in D^c}|x-z|+|y-z|\leq\rho\}} \inf_{t\in[0,1]} V_1(x,y,t) < \delta$

3.3.2 Uniform moderate deviations principle

Proposition 3.3.2. *Fixed $T > 0$ and $x \in \mathbb{R}^d$, let $F \subset \mathbb{D}([0,T],\mathbb{R}^d)$ be closed and $G \subset \mathbb{D}([0,T],\mathbb{R}^d)$ open with respect to the Skorokhod topology. Then we have*

$$\begin{aligned} &a)\ \limsup_{\varepsilon\to;y\to x} \varepsilon^\alpha \ln \bar{\mathbb{P}}(Y^{\varepsilon,y} \in F) \leq -\inf_{f\in F} \tilde{\mathbb{I}}_1(f)_{x,T} \\ &b)\ \liminf_{\varepsilon\to 0;y\to x} \varepsilon^\alpha \ln \bar{\mathbb{P}}(Y^{\varepsilon,y} \in G) \geq -\inf_{g\in G} \tilde{\mathbb{I}}_1(g)_{x,T}. \end{aligned}$$

Proof. In order to use **Theorem D.1.1**, we consider $(x_\varepsilon)_{\varepsilon>0} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ such that $x_\varepsilon \to x$ for the usual topology of $\mathbb{R}^d$. We fix $\delta > 0$ and show that

$$\limsup_{\varepsilon\to 0} \varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(d_{J_1}(Y^{\varepsilon,x_\varepsilon}, Y^{\varepsilon,x}) > \delta\Big) = -\infty. \tag{3.3.1}$$

By definition of the J_1 metric in (1.2.3), for $\varphi, \psi \in \mathbb{D}([0,T],\mathbb{R}^d)$ it follows

$$d_{J_1}(\varphi,\psi) \leq \sup_{t\in[0,T]} |\varphi(t) - \psi(t)|$$

and therefore

$$\bar{\mathbb{P}}\Big(d_{J_1}(Y^{\varepsilon,x_\varepsilon}, Y^{\varepsilon,x}) > \delta\Big) \leq \bar{\mathbb{P}}\Big(\sup_{t\in[0,T]} |Y_t^{\varepsilon,x_\varepsilon} - Y_t^{\varepsilon,x}| > \delta\Big).$$

For every $\varepsilon > 0$ and $t \geq 0$, let

$$\begin{aligned} u(t;x) &= X_t^{0,x} = x - \int_0^t \nabla U(u(s;x))ds, \\ u(t;x_\varepsilon) &= X_t^{0,x_\varepsilon} = x_\varepsilon - \int_0^t \nabla U(u(s;x_\varepsilon))ds. \end{aligned}$$

Due to (1.1.1), *Gronwall's lemma* implies

$$\sup_{t\geq 0} |u(t;x) - u(t;x_\varepsilon)| \leq |x - x_\varepsilon|^2 e^{-2\eta T}.$$

Therefore, we conclude

$$\Big\{\sup_{t\in[0,T]}|Y_t^{\varepsilon,x_\varepsilon}-Y_t^{\varepsilon,x}|>\delta\Big\}=\bar{\mathbb{P}}\Big\{\sup_{t\in[0,T]}|u(t;x)-u(t;x_\varepsilon)|>\delta a(\varepsilon)\Big\}.$$

We choose $\varepsilon_0>0$ such that for every $\varepsilon<\varepsilon_0$ we have $a(\varepsilon)\le 1$ and $|x_\varepsilon-x|<\delta$ This finishes the proof of (3.3.1). □

It follows, as a corollary, the large deviations uniform in compact sets of initial states for $(Y^{\varepsilon,x})_{\varepsilon>0}$. The proof is analogous to the proof of **Corollary 2.3.2** and we omit it in this chapter.

Corollary 3.3.2. *Let $K\subset\mathbb{R}^d$ be compact, $F\subset\mathbb{D}([0,T],\mathbb{R}^d)$ closed, $G\subset\mathbb{D}([0,T],\mathbb{R}^d)$ open and $x\in\mathbb{R}^d$. Then we have*

$$a)\ \lim\varepsilon\to 0\sup_{y\in K}\varepsilon^\alpha\ln\bar{\mathbb{P}}(Y^{\varepsilon,y}\in F)\le-\inf_{y\in K,f\in F}\tilde{\mathbb{I}}_1(f)_{y,T},$$
$$b)\ \liminf_{\varepsilon\to0}\inf_{y\in K}\varepsilon^\alpha\ln\bar{\mathbb{P}}(Y^{\varepsilon,y}\in G)\ge-\inf_{y\in K,g\in G}\tilde{\mathbb{I}}_1(g)_{y,T}.$$

3.3.3 Proof of Theorem 1.3.3

1. We start with the proof of the upper bound.

   Claim 3.3.1. *For any $\delta>0$, there exists $s>0$ and $\varepsilon_0>0$ such that, for every $\varepsilon<\varepsilon_0$, we have*

   $$\inf_{x\in D}\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x)\le s)\ge e^{-\frac{\bar{V}_1+\frac{\delta}{2}}{\varepsilon^\alpha}}.$$

   Proof. As in in the proof of **Theorem 2.3.2**, we can show that, for any $\delta>0$, there exists $s>0$ and $\varepsilon_0>0$ such that, for every $\varepsilon<\varepsilon_0$, we have

   $$\inf_{x\in D}\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x)\le s)\ge e^{-\frac{\bar{V}_1+\frac{\delta}{2}}{\varepsilon^\alpha}}.$$

   As in **Lemma 2.3.1**, but with ε replaced by ε^α, we can show that, given $x\in\mathbb{R}^d$ and for every $\delta>0$, there exist $s_0>0$ and $\rho>0$ such that

   $$\liminf_{\varepsilon\to0}\varepsilon^\alpha\ln\inf_{|x|\le\rho}\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x)\le s_0)>-(\bar{V}_1+\frac{\delta}{4}).$$

   We consider $\rho>0$ small enough such that $\mathrm{cl}(B_\rho(0))\subset D$. We define, for every $x\in D$ and $\varepsilon>0$ the random variable

   $$\tilde{\tau}_\rho^\varepsilon(x):=\inf\{t\ge0\mid |Y_t^{\varepsilon,x}|\le\rho\quad\text{or}\quad Y_t^{\varepsilon,x}\notin D\}.$$

Replacing ε by ε^α in the proof of **Lemma 2.3.2**, for the chosen value of $\rho > 0$ there exists a time $s_1 > 0$ such that

$$\limsup_{\varepsilon \to 0} \varepsilon^\alpha \ln \sup_{x \in D} \bar{\mathbb{P}}(\tilde{\tau}_\rho^\varepsilon(x) > s_1) = -\infty.$$

This implies for any $r > 0$ the existence of $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, we have

$$\varepsilon^\alpha \ln \sup_{x \in D} \bar{\mathbb{P}}(\tilde{\tau}_\rho^\varepsilon(x) > s_1) < -r.$$

In addition, we choose $\varepsilon_0 > 0$ sufficiently small such that for $\varepsilon < \varepsilon_0$ the inequality $1 - e^{-\frac{r}{\varepsilon^\alpha}} > e^{-\frac{\delta}{4\varepsilon^\alpha}}$ holds.
We note that on the event $\{\tilde{\tau}_\rho^\varepsilon(x) < \tilde{\sigma}^\varepsilon(x)\}$ the following identity is valid,

$$\tilde{\sigma}^\varepsilon(x) = \tilde{\tau}_\rho^\varepsilon(x) + \tilde{\sigma}^\varepsilon(X^{\varepsilon,x}_{\tilde{\tau}_\rho^\varepsilon}(x)) \circ \Theta_{\tilde{\tau}_\rho^\varepsilon(x)},$$

where Θ_s is the usual shift on the path space $\mathbb{D}([0,\infty), \mathbb{R}^d)$ by a fixed time $s \geq 0$. As in **Proposition 2.3.1** the strong Markov property of the jump diffusion $(X^{\varepsilon,x})_{\varepsilon>0}$ (see **Proposition 1.1.1**) implies for $\varepsilon < \varepsilon_0$ and fixed $x \in D$,

$$\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) \leq s_0 + s_1) \geq e^{-\frac{\bar{V}_1 + \frac{\delta}{2}}{\varepsilon^\alpha}}.$$

Setting $s = s_0 + s_1$ we have proved the claim. $\square$

Claim 3.3.2. We have for every $\delta > 0$ and $x \in D$

$$\lim_{\varepsilon \to 0} \bar{\mathbb{P}}\Big(\tilde{\sigma}^\varepsilon(x) \leq e^{\frac{\bar{V}_1 + \delta}{\varepsilon^\alpha}}\Big) = 1.$$

Proof. We proceed as in **Theorem 2.3.2**. We write

$$q^\varepsilon = \inf_{x \in D} \bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) \leq s).$$

Using the previous claim we have that $q^\varepsilon > 0$ for $\varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ is given in the previous result. For $k \in \mathbb{N}$, $x \in D$, we consider the event $\{\tilde{\sigma}^\varepsilon(x) > ks\}$. Conditioning, we derive

$$\begin{aligned}
\bar{\mathbb{P}}\Big(\tilde{\sigma}^\varepsilon(x) > (k+1)s\Big) &= \Big(1 - \bar{\mathbb{P}}\Big(\tilde{\sigma}^\varepsilon(x) \leq (k+1)T | \tilde{\sigma}^\varepsilon(x) > ks\Big)\Big) \bar{\mathbb{P}}\Big(\tilde{\sigma}^\varepsilon(x) > ks\Big) \\
&\leq (1 - q^\varepsilon) \bar{\mathbb{P}}\Big(\tilde{\sigma}^\varepsilon(x) > ks\Big).
\end{aligned}$$

By recursion in $k \in \mathbb{N}$, for $\varepsilon < \varepsilon_0$, we obtain

$$\sup_{x \in D} \bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) > ks) \leq (1 - q^\varepsilon)^k,$$

which implies the following bound

$$\sup_{x\in D}\bar{\mathbb{E}}[\tilde{\sigma}^\varepsilon(x)] \le \sup_{x\in D}\sum_{k=0}^{+\infty}\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) > ks) \le s\sum_{k=0}^{+\infty}(1-q^\varepsilon)^k = \frac{s}{q^\varepsilon}.$$

The estimate $q^\varepsilon \ge e^{-\frac{\bar{V}_1+\frac{\delta}{2}}{\varepsilon^\alpha}}$ yields

$$\sup_{x\in D}\bar{\mathbb{E}}[\tilde{\sigma}^\varepsilon(x)] \le s e^{\frac{\bar{V}_1+\frac{\delta}{2}}{\varepsilon^\alpha}}.$$

From *Chebyshev's inequality* we conclude, for all $x \in D$ and $\varepsilon < \varepsilon_0$,

$$\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) \ge e^{\frac{\bar{V}_1+\delta}{\varepsilon^\alpha}}) \le e^{-\frac{\bar{V}_1+\delta}{\varepsilon^\alpha}}\bar{\mathbb{E}}[\tilde{\sigma}^\varepsilon(x)] \le s e^{-\frac{\delta}{2\varepsilon^\alpha}}.$$

Letting $\varepsilon \to 0$ we prove the upper bound.

□

2. Before the proof of the lower bound, we present two claims that are used in the sequel.

Claim 3.3.3. For every $x \in D$ and $\rho > 0$ such that $\mathrm{cl}(B_\rho(0)) \subset D$, we have

$$\lim_{\varepsilon\to 0}\bar{\mathbb{P}}\Big(Y^{\varepsilon,x}_{\tilde{\tau}^\varepsilon_\rho(x)} \in \mathrm{cl}(B_\rho(0))\Big) = 1.$$

Proof. Fix $x \in D$ and $\rho > 0$ such that $\mathrm{cl}(B_\rho(0)) \subset D$. The fact $\tau^\varepsilon_\rho(x) < \infty$, $\bar{\mathbb{P}}$-a.s. implies the following inclusion of events

$$\Big\{Y^{\varepsilon,x}_{\tilde{\tau}^\varepsilon_\rho(x)} \in \mathrm{cl}(B_\rho(0))\Big\} \supset \Big\{Y^{\varepsilon,x}_t \in \mathrm{cl}(B_\rho(0)) \quad \text{for all } t \ge 0\Big\}.$$

Therefore,

$$\begin{aligned}\bar{\mathbb{P}}\Big(Y^{\varepsilon,x}_{\tilde{\tau}^\varepsilon_\rho(x)} \in \mathrm{cl}(B_\rho(0))\Big) &\le \bar{\mathbb{P}}\Big(\sup_{t\ge 0}|Y^{\varepsilon,x}_t|^2 > \rho^2\Big)\\ &\le \frac{1}{\rho^2}\bar{\mathbb{E}}\Big[\sup_{t\ge 0}|Y^{\varepsilon,x}_t|^2\Big]. \end{aligned} \tag{3.3.2}$$

We drop the dependence in the initial condition $x \in D$.
Using *Itô's formula* and observing that **Condition 1.1.1** holds, we derive, for every $t \ge 0$,

$$\begin{aligned}|X^\varepsilon_t - X^0_t|^2 + 2\eta\int_0^t |X^\varepsilon_s - X^0_s|ds \le\ & 2\Big|\int_0^s\int_{\mathbb{R}^d}\langle\varepsilon z, X^\varepsilon_{s-} - X^0_s\rangle\tilde{N}^{\frac{1}{\varepsilon}}(ds,dz)\Big|\\ &+ \varepsilon^2\Big|\int_0^t\int_{\mathbb{R}^d}|z|^2\tilde{N}^{\frac{1}{\varepsilon}}(ds,dz)\Big|\\ &+ \varepsilon\int_0^t\int_{\mathbb{R}^d}|z|^2\nu(dz)ds. \end{aligned} \tag{3.3.3}$$

We write $c_\nu^2 = \int_{\mathbb{R}^d} |z|^2 \nu(dz)$ and for every $t \geq 0$,

$$M_t^1 = \int_0^t \int_{\mathbb{R}^d} \varepsilon^2 |z|^2 \tilde{N}^{\frac{1}{\varepsilon}}(ds, dz),$$
$$M_t^2 = \int_0^t \int_{\mathbb{R}^d} \langle \varepsilon z, X_{s-}^\varepsilon - X_s^0 \rangle \tilde{N}^{\frac{1}{\varepsilon}}(ds, dz).$$

We observe, for every $T \geq 0$, that

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\leq s\leq T} |M_s^1|] &\leq 2\varepsilon \bar{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{R}^d} |z|^2 \nu(dz) ds\Big] \\ &\leq 2\varepsilon c_\nu^2 T.\end{aligned}$$

Applying *Burkholder-Davis-Gundy's inequalities*, we conclude, for some $C > 0$ and for every $\delta > 0$, that

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\leq s\leq T} |M_s^2|] &\leq 2\varepsilon C \bar{\mathbb{E}}\Big[\Big(\int_0^T \int_{\mathbb{R}^d} |\langle \varepsilon z, X_s^\varepsilon - X_s^0\rangle|^2 N^{\frac{1}{\varepsilon}}(ds, dz)\Big)^{1/2}\Big] \\ &\leq \frac{C\varepsilon}{\delta} \bar{\mathbb{E}}[\sup_{0\leq s\leq T} |X_s^\varepsilon - X_s^0|^2] + C\varepsilon\delta \int_0^T \int_{\mathbb{R}^d} |z|^2 \nu(dz) ds.\end{aligned}$$

Taking $\delta = 2C\varepsilon$, (3.3.3) implies, for every $\varepsilon > 0$ and $T > 0$,

$$\frac{1}{2}\bar{\mathbb{E}}[\sup_{0\leq s\leq T} |Y_s^\varepsilon|^2] + \eta \int_0^T \bar{\mathbb{E}}[\sup_{0\leq r\leq s} |Y_r^\varepsilon|^2] ds \leq \frac{\varepsilon}{a^2(\varepsilon)} c_\nu^2 T (2 + 2C^2\varepsilon).$$

Gronwall's lemma yields, for every $T > 0$,

$$\bar{\mathbb{E}}\Big[\sup_{0\leq s\leq T} |Y_s^\varepsilon|^2\Big] \leq \frac{\varepsilon}{a^2(\varepsilon)} 4c_\nu^2 T (1 + 1C^2\varepsilon) e^{-2\eta T}.$$

From (3.3.2) and monotone convergence theorem we have

$$\begin{aligned}\bar{\mathbb{P}}\Big(Y_{\tilde{\tau}_\rho^\varepsilon(x)}^{\varepsilon,x} \in \mathrm{cl}(B_\rho(0))\Big) &\leq \frac{1}{\rho^2} \bar{\mathbb{E}}\Big[\sup_{t\geq 0} |Y_t^{\varepsilon,x}|^2\Big] \\ &\leq \frac{\varepsilon}{a^2(\varepsilon)} 4c_\nu^2 \sup_{T\geq 0} T(1 + C^2\varepsilon) e^{-2\eta T} \\ &\leq \frac{\varepsilon}{a^2(\varepsilon)} 4c_\nu^2 \frac{1}{2\eta} e^{-1} (1 + C^2\varepsilon) \\ &\to 0 \quad \text{as } \varepsilon \to 0,\end{aligned}$$

and the result follows. □

Claim 3.3.4. For every $T > 0$, $x \in D$, $\rho > 0$ and $c > 0$ we have

$$\limsup_{\varepsilon\to 0} \varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T} |Y_t^{\varepsilon,x}| \geq \rho\Big) \leq -c.$$

Proof. We proceed as in the proof of **Theorem 2.3.1**. Fix $T > 0$, $x \in D$, $\rho > 0$ and $c > 0$. Fixed $\varepsilon_0 > 0$ small enough such that for every $\varepsilon < \varepsilon_0$ we have

$$a(\varepsilon) \leq \frac{2T\nu(\mathbb{R}^d)}{\rho},$$

it follows that

$$\begin{aligned}\bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T}|Y_t^{\varepsilon,x}| \geq \rho\Big) &= \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T}|X_t^{\varepsilon,x} - X_t^{0,x}| \geq \rho a(\varepsilon)\Big)\\ &\leq \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T}|X_t^{\varepsilon,x} - w(t;x)|^2 \geq \frac{\rho^2 a^2(\varepsilon)}{4}\Big) + 1,\end{aligned}$$

where the function $w(.;x)$ is defined, for every $t \geq 0$,

$$w(t;x) = X_t^{0,x} - \int_0^t \int_{\mathbb{R}^d} z\nu(dz)ds.$$

Recalling $(\varepsilon W_i)_{i\in\mathbb{N}}$ the jumps and $(\varepsilon T_i)_{i\in\mathbb{N}}$ the jumping times of $(X_t^{\varepsilon,x})_{t\geq 0}$ (**Remark 2.3.1**), we have, for $\varepsilon T_n \leq t \leq \varepsilon T_{n+1}$,

$$X_t^{\varepsilon,x} = w\Big(t - \varepsilon T_n; X_{\varepsilon T_n}^{\varepsilon,x}\Big) + \varepsilon W_{n+1}\mathbf{1}_{\{t=\varepsilon T_{n+1}\}}.$$

As it was seen before in the proof of **Theorem 2.3.1**, we derive the following recurrence relation, for all $n \in \mathbb{N}$, $x, y \in \mathbb{R}^d$,

$$|X_{\varepsilon T_{n+1}}^{\varepsilon,x} - w(\varepsilon T_n; y)|^2 \leq q_\varepsilon |X_{\varepsilon T_n}^{\varepsilon,x} - w(\varepsilon T_n; y)|^2 + 2\varepsilon^2 |W_{n+1}|^2,$$

where $q_\varepsilon = 2e^{-2\eta\theta}$, for some $\theta > 0$ arbitrarily fixed such that $q_\varepsilon < 1$.

Setting $x = y$, in an analogous way to what was done in the proof of **Theorem 2.3.1** we conclude, due to *Kolomogorov 3 series theorem* (**Proposition C.1.9**), for all $x \in \mathbb{R}^d$, $\bar{\mathbb{P}}$-a.s.

$$\begin{aligned}\sup_{t\geq 0}|X_t^{\varepsilon,x} - w(t;x)|^2 &= \sup_{n\in\mathbb{N}}|X_{\varepsilon T_n}^{\varepsilon,x} - w(\varepsilon T_n;x)|^2\\ &\leq 2\varepsilon S^2,\end{aligned}$$

where

$$S^2 := \lim_{n\to\infty}\sum_{i=1}^n q_\varepsilon^{n-i}|W_i|^2 =^d \lim_{n\to\infty}\sum_{i=1}^n q_\varepsilon^i |W_i|^2,$$

since $(W_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with law $\frac{\nu}{\nu(\mathbb{R}^d)}$.

Combining the same arguments used to derive (2.3.10) and (2.3.11) and the fact

$\ln(x+y) \leq \ln x + \ln y + \ln 2$ we conclude, for $\varepsilon_0 > 0$ enough such that $a(\varepsilon) \leq \frac{2T\nu(\mathbb{R}^d)}{\rho}$

$$
\begin{aligned}
&\varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{t\geq 0} |X_t^{\varepsilon,x}| \geq \rho a(\varepsilon)\Big) \\
&\leq \varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{t\geq 0} |X_t^{\varepsilon,x} - w(t;x)|^2 \geq \frac{\rho a^2(\varepsilon)}{4}\Big) + \varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{t\geq 0} |w(t;x) - X_t^{0,x}| \geq \frac{\rho a^2(\varepsilon)}{4}\Big) \\
&\leq -\varepsilon^\alpha (C(\rho,\varepsilon)\varepsilon^{-1})^\alpha + \varepsilon^\alpha + \varepsilon^\alpha \ln 2 \\
&\leq -C^\alpha(\rho,\varepsilon) + \varepsilon^\alpha + \varepsilon^\alpha \ln 2, \qquad (3.3.4)
\end{aligned}
$$

where (cf. with (2.3.9))

$$
C(\rho,\varepsilon) := \frac{\rho a(\varepsilon)}{2\sqrt{2}} \sqrt{\frac{1-\sqrt{q_\varepsilon}}{q_\varepsilon}}. \qquad (3.3.5)
$$

Therefore, since $a(\varepsilon) = \varepsilon^{\frac{1-\alpha}{2}}$,

$$
\varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{t\geq 0} |X_t^{\varepsilon,x}| \geq \rho a(\varepsilon)\Big) \leq \frac{\rho^\alpha \varepsilon^{\frac{\alpha(1-\alpha)}{2}}}{(2\sqrt{2})^\alpha} \Big(\frac{1-\sqrt{q_\varepsilon}}{q_\varepsilon}\Big)^{\frac{\alpha}{2}}
$$

For the fixed value of $c > 0$, we solve the equation for q_ε,

$$
\frac{\rho^\alpha}{(2\sqrt{2})^\alpha} \Big(\frac{1-\sqrt{q_\varepsilon}}{q_\varepsilon}\Big)^{\frac{\alpha}{2}} = \varepsilon^{-\frac{\alpha(1-\alpha)}{2}} c.
$$

and we obtain

$$
q_\varepsilon^\pm = \rho^2 \frac{-1 \pm \sqrt{1 + \frac{32c^{2/\alpha}}{\rho^2} \varepsilon^{-\frac{\alpha(1-\alpha)}{\alpha}}}}{16c^{2/\alpha} \varepsilon^{-\frac{\alpha(1-\alpha)}{\alpha}}}.
$$

In abuse of notation, we write $q_\varepsilon = q_\varepsilon^+$ since $q_\varepsilon > 0$. Since $q_\varepsilon = 2e^{-2\theta}$, we write

$$
\theta = \theta(\varepsilon) = -\frac{1}{2} \ln\Big(\frac{q_\varepsilon}{2}\Big).
$$

We observe that $\lim_{\varepsilon\to 0} \theta(\varepsilon) = \infty$. Plugging the expression of q_ε in (3.3.5) and consequently in (3.3.4) it follows

$$
\limsup_{\varepsilon\to 0} \varepsilon^\alpha \ln \bar{\mathbb{P}}\Big(\sup_{0\leq t\leq T} |Y_t^{\varepsilon,x}| \geq \rho\Big) \leq -c.
$$

□

3. We prove next the upper bound. The proof is analogous to the proof of **Theorem 2.3.2**.

Claim 3.3.5. For any $\delta > 0$, $x \in D$, we have

$$\lim_{\varepsilon \to 0} \bar{\mathbb{P}}\Big(\tilde{\sigma}^\varepsilon(x) \geq e^{\frac{\bar{V}_1 - \delta}{\varepsilon^\alpha}}\Big) = 1$$

Proof. Choose $\rho > 0$ such that $\bar{B}_\rho(0) \subset D$. We use the same Markov chain defined in **Theorem 2.3.3**. Recursively, for $x \in D$, $k \in \mathbb{N}$, let

$$\begin{aligned}
&\tilde{\zeta}_0^x := 0 \\
&\tilde{\tau}_k^x := \inf\{t \geq \tilde{\zeta}_x^k : Y_t^{\varepsilon,x} \in \mathrm{cl}(B_\rho(0)) \cup D^c\} \\
&\tilde{\zeta}_{k+1}^x := \begin{cases} \infty & \text{if } Y_{\tilde{\tau}_k^x}^{\varepsilon,x} \in D^c \\ \inf\{t \geq \tilde{\tau}_k^x : Y_t^{\varepsilon,x} \in (\mathrm{cl}(B_\rho(0))^c\} & \text{if } Y_{\tilde{\tau}_k^x}^{\varepsilon,x} \in \mathrm{cl}(B_\rho(0)). \end{cases}
\end{aligned}$$

$(Y_{\tilde{\tau}_k^x}^{\varepsilon,x})$ is a Markov chain, with the convention $Y_{\tilde{\tau}_k^x}^{\varepsilon,x} := Y_{\tilde{\sigma}^\varepsilon(x)}^{\varepsilon,x}$ if $\tilde{\tau}_k^x = \infty$.

Fix $\delta > 0$. Using an analogous result of **Lemma 2.3.5** but with ε replaced by ε^α, there exists $\rho_0 > 0$ such that, for any $0 < \rho < \rho_0$, $k \in \mathbb{N}$, we have

$$\limsup_{\varepsilon \to 0} \varepsilon^\alpha \ln \sup_{|x| \leq 2\rho} \bar{\mathbb{P}}(Y_{\tilde{\tau}_\rho^x}^{\varepsilon,x} \in D^c) \leq -\bar{V}_1 + \frac{\delta}{2}.$$

Fix now $\rho < \rho_0$ and write $c = \bar{V}_1 + \dfrac{\delta}{2}$. For given $T_0 > 0$, **Claim 3.3.4** yields

$$\bar{\mathbb{P}}(\sup_{0 \leq t \leq T_0} |Y_t^{\varepsilon,x}| \geq \rho) \leq e^{-\frac{\bar{V}_1 - \frac{\delta}{2}}{\varepsilon^\alpha}}.$$

Strong Markov property implies that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \leq \varepsilon_0$,

$$\sup_{x \in D} \bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) = \tilde{\tau}_x^k) \leq \sup_{|y| \leq 2\rho} \bar{\mathbb{P}}(\tilde{X}_{\tau_\rho^\varepsilon(x)}^{\varepsilon,y} \in D^c) \leq e^{-\frac{\bar{V}_1 - \frac{\delta}{2}}{\varepsilon^\alpha}}$$

and

$$\sup_{x \in D} \bar{\mathbb{P}}(\tilde{\zeta}_k^x - \tilde{\tau}_{k-1}^x \leq T_0) \leq \bar{\mathbb{P}}(\sup_{0 \leq t \leq T_0} |Y_t^{\varepsilon,x}| \geq \rho) \leq e^{-\frac{\bar{V}_1 - \frac{\delta}{2}}{\varepsilon^\alpha}}.$$

Fix $n \in \mathbb{N}$. For any $x \in D$, the following inclusion holds:

$$\{\tilde{\sigma}^\varepsilon(x) \leq kT_0\} \subset \{\tilde{\sigma}^\varepsilon(x) = \tilde{\tau}_0^x\} \cup \bigcup_{m=1}^k \{\tilde{\sigma}^\varepsilon(x) = \tilde{\tau}_m^x\} \cup \{\tilde{\nu}_m^x - \tilde{\tau}_{m-1}^x \leq T_0\}.$$

Hence, as in **Theorem 2.3.2**, we have, for any $k \in \mathbb{N}$ and $x \in D$,

$$\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) \leq kT_0) \leq \bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x)) = \tilde{\tau}_x^0) + 2ke^{-\frac{\bar{V}_1 - \frac{\delta}{2}}{\varepsilon^\alpha}}.$$

Take $k := [\frac{1}{T_0} e^{\frac{\bar{V_1} - \delta}{\varepsilon}}] + 1$.
The last estimate combined with **Claim 3.3.3** yield, for all $x \in D$,

$$\begin{aligned}\bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) \leq e^{\frac{\bar{V_1} - \delta}{\varepsilon^\alpha}}) &\leq \bar{\mathbb{P}}(\tilde{\sigma}^\varepsilon(x) \leq kT_0) \\ &\leq \bar{\mathbb{P}}(Y^{\varepsilon,x}_{\tilde{\tau}^x_\rho} \notin \bar{B}_\rho(0)) + \frac{4}{T_0} e^{-\frac{\delta}{2\varepsilon^\alpha}} \to 0.\end{aligned}$$

The lower bound for $\bar{\mathbb{E}}[\tilde{\sigma}^\varepsilon(x)]$ follows from *Chebyshev's inequality.*

The case $\bar{V} = 0$ follows with analogous considerations from the ones of the proof of **Theorem 2.3.3**. $\square$

Chapter 4

The small noise limit for a coupled FBSDE system with jumps

4.1 Motivation and the probabilistic setup

As was pointed out in the introduction, it is our intent to study the vanishing viscosity limit of a certain class of partial-integral differential equations (PIDEs for short) using a forward backward system of stochastic differential equations (FBSDEs for short) with jumps. FBSDEs give stochastic representations to the solutions of semilinar PDEs. We use this representation to study the convergence of the solutions of the associated PDE when the diffusive term is affected by a parameter that vanishes. In this class of PDEs we include the fractal Burgers equation that was discussed in the introduction. These equations form a simple model for the velocity of a compressible fluid that has a fractional diffusive behaviour affected by an external force that allows nonlocal sources of interaction. Furthermore, we present, under more restrictive assumptions, a large deviations principle for the laws of the the forward and backward processes that solve the FBSDE system.

We introduce the necessary probabilistic setup for our work.

Fix $T > 0$, $0 < T' < T$ and $t \in [T', T]$. Let $C_0([t,T], \mathbb{R}^d)$ be the space of the continuous functions $f : [t,T] \longrightarrow \mathbb{R}^d$ such that $f(t) = 0$. $\mathbb{M}_{t,T}$ denotes the space of the locally finite measures defined on the Borel sets of $[t,T] \times \mathbb{R}^d$. We fix a σ-finite Lévy measure ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, i.e. with the property $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(\mathrm{d}z) < \infty$.

In $C_0([t,T], \mathbb{R}^d)$ we consider the coordinate map

$$\begin{aligned} B : C_0([t,T], \mathbb{R}^d) &\longrightarrow C_0([t,T], \mathbb{R}^d), \\ B(\omega)(s) &:= \omega(s), \quad s \in [t,T]. \end{aligned}$$

We denote $\mathbb{V} := C_0([t,T], \mathbb{R}^d) \times \mathbb{M}_{t,T}$. We denote by $\bar{\mathbb{M}}_{t,T}$ the space of locally finite measures defined in the Borel sets of the product space $[t,T] \times \mathbb{R}^d \times [0,\infty)$. Analogoulsy to what was pointed out in the first section of **Chapter 1** the purpose of such space is the following: given a Poisson random measure, the first component registers the time of the jumps, the second one is the space of the jumps and the third one registers their frequency. We consider the canonical map

$$\begin{aligned} \bar{N} : \bar{\mathbb{M}}_{t,T} &\longrightarrow \bar{\mathbb{M}}_{t,T}, \\ \bar{N}(\bar{m}) &:= \bar{m}. \end{aligned}$$

We denote $\bar{\mathbb{V}} := C_0([t,T],\mathbb{R}^d) \times \bar{\mathbb{M}}_{t,T}$. *Ikeda and Watanabe (1981)*-p.77- **Theorem 6.3** implies that there exists a unique probability measure $\bar{\mathbb{P}}$ defined on the Borel measurable space $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}))$ such that B is a Brownian motion with values in $\mathbb{R}^d$ and $\bar{N}$ is an independent Poisson random measure with intensity $dr \otimes \nu \otimes du$, where du stands for the Lebesgue measure on the Borel sets of the third component of the product space $[t,T] \times \mathbb{R}^d \times [0,\infty)$. In a similar way of what was presented in the last section, given $\theta > 0$, if $N^{\frac{1}{\theta}}$ is the Poisson random measure defined on $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}))$ and $\tilde{N}^{\frac{1}{\theta}}$ its compensated version with compensator $\frac{1}{\theta} dr \otimes \nu$, we have the following representation of $N^{\frac{1}{\theta}}$ in terms of the measure $\bar{N}$. Given $s \in [t,T]$, $U \in \mathcal{B}(\mathbb{R}^d)$,

$$N^{\frac{1}{\theta}}([t,s] \times U) := \int_t^s \int_U \int_0^\infty \mathbf{1}_{[0,\frac{1}{\theta}]}(u) \bar{N}(dr,dz,du). \tag{4.1.1}$$

We use the identification $N = N^1$ due to (4.1.1). Consider the filtration $(\mathcal{G}_s)_{t \leq s \leq T}$ defined on $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}), \bar{\mathbb{P}})$ generated by the two processes B and $\bar{N}$, i.e. for all $s \in [t,T]$,

$$\mathcal{G}_s := \sigma\Big\{ \bar{N}((t,u] \times A \times C), B_u \mid t \leq u \leq s, A \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{B}([0,\infty)) \Big\}.$$

We denote by $(\bar{\mathcal{G}}_s)_{s \in [t,T]}$ the completion of the filtration $(\mathcal{G}_s)_{s \in [t,T]}$ under $\bar{\mathbb{P}}$. Let $\mathcal{P}$ be the predictable σ-field on $\bar{\mathbb{V}} \times [t,T]$ and $\bar{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$.

Fix $x \in \mathbb{R}^d$ and the Borel measurable functions

$$\begin{aligned} b &: \mathbb{R}^d \longrightarrow \mathbb{R}^d, \\ \beta &: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d. \end{aligned}$$

Let us impose the following Lipschitz and growth conditions on b and β.

Condition 4.1.1. There exists $K > 0$ such that, for all $y, y_1 \in \mathbb{R}^d$,

i) $|b(y) - b(y_1)|^2 + \displaystyle\int_{\mathbb{R}^d} |\beta(y,z) - \beta(y_1,z)| \nu(dz) \leq K|y - y_1|^2$,

ii) $|b(y)|^2 + \displaystyle\int_{\mathbb{R}^d} |\beta(y,z)|^2 \nu(dz) \leq K(1 + |y|^2)$.

For every $\varepsilon > 0$ and $x \in \mathbb{R}^d$, under **Condition 4.1.1** and due to *Applebaum (2009)*- p. 367- **Theorem 6.2.3** there exists a unique adapted solution $(X_t^{\varepsilon,x})_{t\in[0,T]}$ defined on $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}), \bar{\mathbb{P}})$ with trajectories in $\mathbb{D}([0,T], \mathbb{R}^d)$ of the following stochastic differential equation

$$X_t^{\varepsilon,x} = x + \int_0^t b(X_s^{\varepsilon,x})ds + \varepsilon \int_0^t \int_{\mathbb{R}^d} \beta(X_{s-}^{\varepsilon,x}, z)\tilde{N}^{\frac{1}{\varepsilon}}(ds, dz), \quad t \in [0,T]. \tag{4.1.2}$$

For every $\varepsilon > 0$ $\mathcal{I}^\varepsilon$ is the generator of $(X_t^{\varepsilon,x})_{t\in[0,T]}$,

$$\mathcal{I}^\varepsilon \varphi(x) = \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}\varphi(x) + \int_{\mathbb{R}^d} \frac{\varphi(x + \varepsilon\beta(x,z)) - \varphi(x) - \langle \varepsilon\beta(x,z), \nabla_x \varphi(x)\rangle}{\varepsilon} \nu(dz),$$

for every $\varphi \in C^1(\mathbb{R}^d)$. We fix the following functions

$$\begin{aligned} &f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{d\times n} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \\ &g : \mathbb{R}^d \longrightarrow \mathbb{R}^n, \end{aligned}$$

and assume that they are smooth with bounded derivatives.
For every $\varepsilon > 0$, we consider the following terminal value problem for a semilinear PIDE of the type

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \mathcal{I}^\varepsilon u^\varepsilon(t,x) + f(t,x,u^\varepsilon(t,x), \nabla_x u^\varepsilon(t,x), u^\varepsilon(t, x + \varepsilon\beta(x,.)) - u^\varepsilon(t,x)) = 0, \\ u^\varepsilon(T,x) = g(x), \quad t \in [0,T], \quad x \in \mathbb{R}^d. \end{cases} \tag{4.1.3}$$

For every $\varepsilon > 0$, since the assumptions on f and g guarantee enough regularity, it is a classical fact (see *Situ (1997)*) that there exists a classical solution $u^\varepsilon \in C^{1,2}([0,T]\times\mathbb{R}^d, \mathbb{R}^n)$, continuously differentiable in time $t \in [0,T]$ and two times continuously differentiable in space $x \in \mathbb{R}^d$ with values in $\mathbb{R}^n$, of (4.1.3), i.e. that solves pointwise (4.1.3).
For every $\varepsilon > 0$ and $t \in [0,T]$ define

$$Y_t^\varepsilon := u^\varepsilon(t, X_t^{\varepsilon,x}),$$

which is the parametrized solution of (4.1.3) under the flow described by the solution of (4.1.2). Since the function u^ε is regular enough in order to use *Itô's formula* for $(Y_t^\varepsilon)_{t\in[0,T]}$

(see **Proposition B.3.1**), we have, for all $s \in [t,T]$,

$$
\begin{aligned}
dY_s^\varepsilon &= du^\varepsilon(s, X_s^\varepsilon) \\
&= \Big(\partial_s u^\varepsilon(s, X_{s-}^\varepsilon)ds \\
&+ \nabla_x u^\varepsilon(s, X_{s-}^\varepsilon) b(X_{s-}^\varepsilon)ds \\
&+ \int_{\mathbb{R}^d} \varepsilon\beta(X_{s-}^\varepsilon, z)\nabla_x u^\varepsilon(t, X_{s-}^\varepsilon)\tilde{N}^{\frac{1}{\varepsilon}}(dz) \\
&+ \int_{\mathbb{R}^d} \frac{u^\varepsilon(s, X_{s-}^\varepsilon + \varepsilon\beta(X_{s-}^\varepsilon, z)) - u^\varepsilon(s, X_{s-}^\varepsilon) - \varepsilon\beta(X_{s-}^\varepsilon, z), \nabla_x u^\varepsilon(s, X_{s-}^\varepsilon)}{\varepsilon}\nu(dz) \\
&+ \int_{\mathbb{R}^d} (u^\varepsilon(s, X_{s-}^\varepsilon + \varepsilon\beta(X_{s-}^\varepsilon, z)) - u^\varepsilon(s, X_{s-}^\varepsilon) - \varepsilon\beta(X_{s-}^\varepsilon, z), \nabla_x u^\varepsilon(s, X_{s-}^\varepsilon))\tilde{N}^{\frac{1}{\varepsilon}}(dz)\Big) \\
&= (\partial_s u(s, X_s^\varepsilon)ds + \mathcal{I}^\varepsilon u^\varepsilon(s, X_s^\varepsilon))ds \\
&+ \int_{\mathbb{R}^d} (u(s, X_{s-}^\varepsilon + \varepsilon\beta(X_{s-}^\varepsilon, z)) - u^\varepsilon(s, X_{s-}^\varepsilon))\tilde{N}^{\frac{1}{\varepsilon}}(dz) \\
&= -f(s, X_s^\varepsilon, \nabla_x u^\varepsilon(s, X_s^\varepsilon), u^\varepsilon(s, X_s^\varepsilon + \varepsilon\beta(X_s^\varepsilon,)) - u^\varepsilon(s, X_{s-}^\varepsilon)) \\
&+ \int_{\mathbb{R}^d} (u^\varepsilon(s, X_{s-}^\varepsilon + \varepsilon\beta(X_{s-}^\varepsilon, z)) - u^\varepsilon(s, X_{s-}^\varepsilon)\tilde{N}^{\frac{1}{\varepsilon}}(dz)\Big).
\end{aligned}
$$

For every $\varepsilon > 0$, if u^ε solves the terminal value problem (4.1.3), for all $s \in [t,T]$ and $z \in \mathbb{R}^d$,

$$
\begin{cases}
Y_s^\varepsilon := u^\varepsilon(s, X_s^\varepsilon), \\
V_s^\varepsilon(z) := u^\varepsilon(s, X_s^\varepsilon + \varepsilon\beta(X_s^\varepsilon, z)) - u^\varepsilon(s, X_s^\varepsilon),
\end{cases}
$$

solve the equation

$$
dY_s^\varepsilon = -f(s, X_s^\varepsilon, Y_s^\varepsilon, \nabla_x u^\varepsilon(s, X_s^\varepsilon), V_s^\varepsilon) + \int_{\mathbb{R}^d} V_{s-}^\varepsilon(z)\tilde{N}^{\frac{1}{\varepsilon}}(dz).
$$

which should be interpreted as the BSDE

$$
Y_s^\varepsilon = g(X_T^\varepsilon) + \int_t^T f(s, X_s^\varepsilon, Y_s^\varepsilon, \nabla_x u^\varepsilon(s, X_s^\varepsilon), V_s^\varepsilon)ds - \int_t^T \int_{\mathbb{R}^d} V_{s-}^\varepsilon(z)\tilde{N}^{\frac{1}{\varepsilon}}(ds, dz). \quad (4.1.4)
$$

We study the convergence, as $\varepsilon \to 0$ of the solutions of PIDEs such as (4.1.3) via more general systems of FBSDEs than the one constituted by (4.1.2) and (4.1.4). This is the content of the following sections.

4.2 Functional setting and existence and uniqueness of solution in a small time interval

Fix $T > 0$. For every $\varepsilon > 0$ we consider the following Borel measurable functions

$$\begin{aligned}
&b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^d,\\
&\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^{d\times d},\\
&f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,\\
&\beta : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d,\\
&\gamma^\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R} \quad \text{and}\\
&g : \mathbb{R}^d \longrightarrow \mathbb{R}^n.
\end{aligned}$$

We illustrated in the introduction and in the previous section the link between decoupled FBSDEs with jumps as the system constituted by (4.1.2) and (4.1.4) and PIDEs such as (4.1.3). Given and $t \in [0,T]$ and $x \in \mathbb{R}^d$ we consider the asymptotic study as $\varepsilon \to 0$ of the following coupled FBSDE system with jumps:

$$\begin{cases}
X_s^{t,x,\varepsilon} &= x + \displaystyle\int_t^s b(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon})dr + \sqrt{\varepsilon}\int_t^s \sigma(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon})dB_r\\
&\quad + \varepsilon \displaystyle\int_t^s \int_{\mathbb{R}^d} \beta(X_{r-}^{t,x,\varepsilon}, z)\tilde{N}^{\frac{1}{\varepsilon}}(dr,dz),\\
Y_s^{t,x,\varepsilon} &= g(X_T^{t,x,\varepsilon}) + \displaystyle\int_s^T f\Big(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon}, Z_r^{t,x,\varepsilon}, \int_{\mathbb{R}^d} V_r^{t,x,\varepsilon}(z)\frac{\gamma^\varepsilon(z)}{\varepsilon}\nu(dz)\Big)dr\\
&\quad - \displaystyle\int_s^T Z_r^{t,x,\varepsilon}dB_r - \int_s^T\int_{\mathbb{R}^d} V_{r-}^{t,x,\varepsilon}(z)\tilde{N}^{\frac{1}{\varepsilon}}(dr,dz), \quad t \le s \le T.
\end{cases} \tag{4.2.1}$$

In the next section the reader can find the link between this FBSDE system and a certain PIDE which generalizes the discussion presented in the last section between FBSDE systems with jumps and PIDEs. The goal of this section is to introduce the necessary functional setting for a result of existence and uniqueness of solution in a small time interval of (4.2.1).

Remark 4.2.1 (About the coefficients).

1. *The solution process*

$$(U_s^{t,x,\varepsilon})_{t\le s\le T} = (X_s^{t,x,\varepsilon}, Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}, V_s^{t,x,\varepsilon})_{t\le s\le T}$$

   takes values in $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n$. The dependence on time t, initial condition $x \in \mathbb{R}^d$ and on the parameter $\varepsilon > 0$ are explicit in the notation.

2. *The structure of the generator f of the BSDE in (4.2.1) links the solution process $(X_s^{t,x,\varepsilon}, Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}, V_s^{t,x,\varepsilon})_{t\le s\le T}$ to the viscosity solutions of the associated PIDE. In the next section we present the PIDE associated to the FBSDE system (4.2.1) and define what is a viscosity solution.*

3. *In the backward equation of (4.2.1), for every $\varepsilon > 0$, the term γ^ε is defined in order to guarantee the convergence, when $\varepsilon \to 0$ of the backward process $(Y_s^\varepsilon)_{s\in[t,T]}$ to the solution of a ordinary differential equation, since for every $\varepsilon > 0$ the compensator of $\tilde{N}^{\frac{1}{\varepsilon}}$ is $\frac{1}{\varepsilon}ds \otimes \nu$.*

For every $\varepsilon > 0$ we write, for all $(x, y, z, k) \in \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^d$,

$$h^\varepsilon(s, x, y, z, k) = f\Big(s, x, y, z, \int_{\mathbb{R}^d} k \frac{\gamma^\varepsilon(k)}{\varepsilon} \nu(dk)\Big). \tag{4.2.2}$$

We impose the following conditions on the coefficients of (4.2.1).

Condition 4.2.2. There exist $K_1, K_2, K_3 > 0$ and a Borel measurable positive function $\kappa : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with the property $\frac{\kappa(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$ such that, for every $\varepsilon > 0$, $s \in [t, T]$, $(x, y, z, k), (\bar{x}, \bar{y}, \bar{z}, \bar{k}) \in \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^d$, we have

i)

$$\begin{aligned}
&|b(s, x, y) - b(s, \bar{x}, \bar{y})|^2 + |\sigma(s, x, y) - \sigma(s, \bar{x}, \bar{y})|^2 \\
&+ |h^\varepsilon(s, x, y, z, k) - h^\varepsilon(s, \bar{x}, \bar{y}, \bar{z}, \bar{k})|^2 + |g(x) - g(\bar{x})|^2 \\
&\leq K_1\Big(|x - \bar{x}|^2 + |y - \bar{y}|^2 + |z - \bar{z}|^2 + \int_{\mathbb{R}^d} |k - \bar{k}|^2 \nu(dk)\Big), \\
&|\beta(x, k) - \beta(\bar{x}, k)| \leq K_1(1 \wedge |k|)|x - \bar{x}|,
\end{aligned}$$

ii)

$$\begin{aligned}
\Big\langle \int_{\mathbb{R}^d} (\beta(x, k) - \beta(\bar{x}, k))\nu(dk), x - \bar{x}\Big\rangle &\leq -K_2|x - \bar{x}|^2, \\
\langle b(s, x, y) - b(s, \bar{x}, y), x - \bar{x}\rangle &\leq -K_2|x - \bar{x}|^2, \\
\langle b(s, x, y) - b(s, x, \bar{y}), y - \bar{y}\rangle &\leq -K_2|y - \bar{y}|^2,
\end{aligned}$$

iii)

$$\begin{aligned}
|b(s, x, y)|^2 &\leq K_3(1 + |x|^2 + |y|^2), \\
|h^\varepsilon(s, x, y, z, k)|^2 &\leq K_3\Big(1 + |x|^2 + |y|^2 + |z|^2 + \int_{\mathbb{R}^d} |k|^2 \nu(dk)\Big), \\
|g(x)| &\leq K_3, \\
|\beta(x, z)| &\leq K_3.
\end{aligned}$$

iv) The matrix σ is non degenerate and bounded.

v) For every $\varepsilon > 0$ the function γ^ε satisfies

$$\gamma^\varepsilon(k) \leq K_3\kappa(\varepsilon)(1 \wedge |k|).$$

In what follows we define the functional spaces that we use in the sequel. Fix $t \in [0,T]$, and $k \in \mathbb{N}_1$ and define

$$\mathcal{S}^2(t,T,\mathbb{R}^k) := \Big\{\varphi : \bar{\mathbb{V}} \times [t,T] \longrightarrow \mathbb{R}^k \mid \varphi \text{ is an adapted càdlàg process such that } \bar{\mathbb{E}}\Big[\sup_{t\le s\le T} |\varphi_s|^2\Big] < \infty\Big\},$$

$$\mathcal{H}^2(t,T,\mathbb{R}^k) := \Big\{\varphi : \bar{\mathbb{V}} \times [t,T] \longrightarrow \mathbb{R}^k \mid \varphi \text{ is an } (\bar{\mathcal{G}}_s)_{s\in[t,T]} - \text{ predictable process such that } \bar{\mathbb{E}}\Big[\int_t^T |\varphi_s|^2 ds\Big] < \infty\Big\} \text{ and}$$

$$\mathcal{H}^2_\nu(t,T,\mathbb{R}^k) := \Big\{K : \bar{\mathbb{V}} \times [t,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^k \mid K \text{ is } \widetilde{\mathcal{P}} \text{ -measurable such that } \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} |K_s(z)|^2 \nu(dz) ds\Big] < \infty\Big\}.$$

Given $\rho \geq 0$, these spaces are naturally complete normed spaces, when endowed with the norms

$$||Y||^2_{\rho,\mathcal{S}^2(t,T,\mathbb{R}^k)} := \bar{\mathbb{E}}\Big[\sup_{t\le s\le T} e^{\rho s}|Y_s|^2\Big],$$
$$||Z||_{\rho,\mathcal{H}^2(t,T,\mathbb{R}^k)} := \bar{\mathbb{E}}\Big[\int_t^T e^{\rho s}|Z_s|^2 ds\Big] \text{ and}$$
$$||V||_{\rho,\mathcal{H}^2_\nu(t,T,\mathbb{R}^k)} := \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} e^{\rho s}|V_s(z)|^2 \nu(dz) ds\Big],$$

We omit the dependence in ρ when $\rho = 0$.

Given $T > 0$ and $t \in [0,T]$ we write

$$\mathcal{M}^2[t,T] := \mathcal{S}^2(t,T,\mathbb{R}^d) \times \mathcal{S}^2(t,T,\mathbb{R}^n) \times \mathcal{H}^2(t,T,\mathbb{R}^{n\times d}) \times \mathcal{H}^2_\nu(t,T,\mathbb{R}^n).$$

$\mathcal{M}^2[t,T]$ is a Banach Space endowed with the norm given by

$$||(X,Y,Z,V)||^2_{\mathcal{M}^2[t,T]} := \bar{\mathbb{E}}\Big[\sup_{t\le s\le T} |X_s|^2\Big] + \bar{\mathbb{E}}\Big[\sup_{t\le s\le T} |Y_s|^2\Big] + \bar{\mathbb{E}}\Big[\int_t^T |Z_s|^2 ds\Big] + \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} |V_s(z)|^2 \nu(dz) ds\Big].$$

We state now a result of existence and uniqueness of solution for the FBSDE (4.2.1).

Theorem 4.2.1 (Existence and uniqueness of solution of (4.2.1)). *Given $T > 0$, $x \in \mathbb{R}^d$, under* ***Condition 4.2.2*** *and for every $0 < \varepsilon \leq 1$, there exists $T' < T$ independent of $\varepsilon > 0$, such that for all $t \in [T', T]$ there exists a unique adapted stochastic process $(X^{t,x,\varepsilon}_s, Y^{t,x,\varepsilon}_s, Z^{t,x,\varepsilon}_s, V^{t,x,\varepsilon}_s) \in \mathcal{S}^2(t,T,\mathbb{R}^d) \times \mathcal{S}^2(t,T,\mathbb{R}^n) \times \mathcal{H}^2(t,T,\mathbb{R}^{n\times d}) \times \mathcal{H}^2_\nu(t,T,\mathbb{R}^n)$ that solves (4.2.1) for all $s \in [t,T]$.*

Remark 4.2.3. *The result we prove is not new and it is well-known in the literature of forward-backward stochastic differential equations. It follows from a typical fixed point argument. We refer the reader to Delong (2013) -**Chapter 3** where the proof of existence and uniqueness of solution for BSDEs with jumps is done in the case the generator of the BSDE is Lipschitz continuous and to Peng and Wu (1999), where the authors prove existence and uniqueness of solution for a coupled FBSDE system with jumps that is parametrized by a family of controls. Another reference is the PhD thesis of Fromm (2014), where the reader can find in **Chapter 2** the proof of existence and uniqueness of solutions in small time intervals for a Brownian fully coupled FBSDE system in detail.*
For these reasons we only sketch the proof and remark that the small time interval where the solution of the FBSDE system is defined is independent of $\varepsilon > 0$ if $0 < \varepsilon \leq 1$.

Proof. Fix $x \in \mathbb{R}^d$, $T > 0$, $t \in [0, T]$ and $\varepsilon > 0$. Given $(X, Y, Z, V) \in \mathcal{M}^2[t, T]$, let $(X_s^\varepsilon)_{s \in [t,T]}$ satisfying for every $s \in [t, T]$,

$$X_s^\varepsilon = x + \int_s^s b(r, X_r, Y_r)dr + \sqrt{\varepsilon} \int_t^s \sigma(r, X_r, Y_r)dB_r + \varepsilon \int_t^s \int_{\mathbb{R}^d} \beta(X_{r-}, z)\tilde{N}^{\frac{1}{\varepsilon}}(dr, dz).$$

For every $\varepsilon > 0$, the function h^ε is defined in (4.2.2).
Due to way h^ε is defined, $(h^\varepsilon(X_s^\varepsilon, Y_s, Z_s, V_s))_{s \in [t,T]} \in \mathcal{H}^2(t, T, \mathbb{R}^n)$.

For every $\varepsilon > 0$ and $s \in [t, T]$ we define

$$\begin{aligned} M_s^\varepsilon &:= \bar{\mathbb{E}}\Big[g(X_T^\varepsilon) + \int_t^T h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr \Big| \tilde{\mathcal{G}}_s\Big], \\ Y_s^\varepsilon &:= \bar{\mathbb{E}}\Big[g(X_T^\varepsilon) + \int_s^T h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr \Big| \tilde{\mathcal{G}}_s\Big]. \end{aligned} \tag{4.2.3}$$

It follows from the definitions that for every $\varepsilon > 0$, $(M_s^\varepsilon)_{s \in [t,T]}$ is an $(\bar{\mathcal{G}}_s)_{s \in [t,T]}$ square integrable martingale and

$$M_t^\varepsilon = \bar{\mathbb{E}}\Big[g(X_T^\varepsilon) + \int_t^T h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr \Big| \tilde{\mathcal{G}}_t\Big] = Y_t^\varepsilon. \tag{4.2.4}$$

Theorem B.3.1 implies that, for every $\varepsilon > 0$, there exists

$$(Z_s^\varepsilon, V_s^\varepsilon)_{s \in [t,T]} \in \mathcal{H}^2(t, T, \mathbb{R}^{n \times d}) \times \mathcal{H}_\nu^2(t, T, \mathbb{R}^n)$$

such that, for every $s \in [t, T]$ we have

$$M_s^\varepsilon = M_t^\varepsilon + \int_t^s Z_r^\varepsilon dB_r + \int_t^s \int_{\mathbb{R}^d} V_{r-}^\varepsilon(z)\tilde{N}^{\frac{1}{\varepsilon}}(dr, dz).$$

Since $g(X_T^\varepsilon)$ is $\bar{\mathcal{G}}_T$-measurable, we conclude

$$M_T^\varepsilon = Y_T^\varepsilon.$$

Hence, for all $s \in [0,T]$, (4.2.3) and (4.2.4) yield

$$
\begin{aligned}
Y_s^\varepsilon &= M_s^\varepsilon - \int_t^s h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr \\
&= Y_t^\varepsilon + \int_t^s Z_r^\varepsilon dB_r + \int_t^s \int_{\mathbb{R}^d} V_{r-}^\varepsilon \tilde{N}^{\frac{1}{\varepsilon}}(dr,dz) - \int_t^s h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr \\
&= g(X_T^\varepsilon) + \int_t^T h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr - \int_t^T Z_r^\varepsilon dB_r - \int_t^T \int_{\mathbb{R}^d} V_{r-}^\varepsilon(z) \tilde{N}^{\frac{1}{\varepsilon}}(dr,dz) \\
&- \int_t^s h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr + \int_t^s Z_r^\varepsilon dB_r + \int_t^s V_{r-}^\varepsilon \tilde{N}^{\frac{1}{\varepsilon}}(dr,dz) \\
&= g(X_T^\varepsilon) + \int_s^T h^\varepsilon(r, X_t^\varepsilon, Y_r, Z_r, V_r)dr - \int_s^T Z_r^\varepsilon dB_r - \int_s^T \int_{\mathbb{R}^d} V_{r-}^\varepsilon(z) \tilde{N}^{\frac{1}{\varepsilon}}(dr,dz).
\end{aligned}
$$

We conclude that $Y^\varepsilon \in \mathcal{S}^2(t,T,\mathbb{R}^n)$.

Therefore, for every $\varepsilon > 0$ and $x \in \mathbb{R}^d$, given $(X,Y,Z,V) \in \mathcal{M}^2[t,T]$ we obtain a unique $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon, V^\varepsilon) \in \mathcal{M}^2[t,T]$ that solves the following equations, for every $s \in [t,T]$

$$
\begin{cases}
X_s^\varepsilon &= x + \int_t^s b(r, X_r, Y_r)dr + \sqrt{\varepsilon}\int_t^s \sigma(r, X_r, Y_r)dB_r + \varepsilon \int_t^s \int_{\mathbb{R}^d} \beta(X_{r-}, z)\tilde{N}^{\frac{1}{\varepsilon}}(dr,dz), \\
Y_s^\varepsilon &= g(X_T^\varepsilon) - \int_t^s h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r)dr + \int_t^s Z_r^\varepsilon dB_r + \int_t^s \int_{\mathbb{R}^d} V_{r-}^\varepsilon d\tilde{N}^{\frac{1}{\varepsilon}}(dr,dz).
\end{cases}
\tag{4.2.5}
$$

In this way, for every $\varepsilon > 0$ we constructed a measurable map

$$
\begin{aligned}
&\Theta^\varepsilon : \mathcal{M}^2[t,T] \longrightarrow \mathcal{M}^2[t,T], \\
&\Theta^\varepsilon(X,Y,Z,V) := (X^\varepsilon, Y^\varepsilon, Z^\varepsilon, V^\varepsilon),
\end{aligned}
$$

where $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon, V^\varepsilon)$ are the unique solution of (4.2.5).
We prove next that , for every $0 < \varepsilon \leq 1$, Θ^ε is a contraction.

Given $\varepsilon > 0$, $(X,Y,Z,V), (\bar{X}, \bar{Y}, \bar{Z}, \bar{V}) \in \mathcal{M}^2[t,T]$ we write

$$
\begin{aligned}
\Theta^\varepsilon(X,Y,Z,V) &:= (X^\varepsilon, Y^\varepsilon, Z^\varepsilon, V^\varepsilon), \\
\Theta^\varepsilon(\bar{X}, \bar{Y}, \bar{Z}, \bar{V}) &:= (\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \bar{Z}^\varepsilon, \bar{V}^\varepsilon).
\end{aligned}
$$

It follows from *Itô's fomula* and *Burkholder-Davis-Gundy's inequalities*, since $0 < \varepsilon \leq 1$, that there exists $C_1 = C_1(K_1) > 0$, with the constant K_1 of **Condition 4.2.2**, such that

$$
\bar{\mathbb{E}}\Big[\sup_{t \leq s \leq T} |X_s^\varepsilon - \bar{X}_s^\varepsilon|^2\Big] \leq C_1(T-t)\Big(\bar{\mathbb{E}}\Big[\sup_{t \leq s \leq T} |X_s - \bar{X}_s|^2\Big] + \bar{\mathbb{E}}\Big[\sup_{t \leq s \leq T} |Y_s - \bar{Y}_s|^2\Big]\Big). \tag{4.2.6}
$$

Itò's formula yields, for every $s \in [t,T]$,

$$
\begin{aligned}
&|Y_s^\varepsilon - \bar{Y}_s^\varepsilon|^2 + \int_s^T |Z_r^\varepsilon - \bar{Z}_r^\varepsilon|^2 dr + \int_t^T \int_{\mathbb{R}^d} |V_r^\varepsilon - \bar{V}_r^\varepsilon|^2 \nu(dz) dr \\
&\leq K_1 |X_T^\varepsilon - \bar{X}_T^\varepsilon|^2 + 2\int_s^T |\langle Y_r^\varepsilon - \bar{Y}_r^\varepsilon, h^\varepsilon(r, X_r^\varepsilon, Y_r, Z_r, V_r) - h^\varepsilon(r, \bar{X}_r^\varepsilon, \bar{Y}_r, \bar{Z}_r, \bar{V}_r)\rangle| dr \\
&+ 2\Big|\int_s^T \langle Y_r^\varepsilon - \bar{Y}_r^\varepsilon, (Z_r^\varepsilon - \bar{Z}_r^\varepsilon) dB_r\rangle\Big| + 2\Big|\int_s^T \int_{\mathbb{R}^d} \langle Y_{r-}^\varepsilon - \bar{Y}_{r-}^\varepsilon, V_{r-}^\varepsilon - \bar{V}_{r-}^\varepsilon\rangle \tilde{N}^{\frac{1}{\varepsilon}}(dr, dz)\Big|.
\end{aligned}
$$

The Lipschitz condition on f (**Condition 4.2.2**), the last inequality, *Burkholder-Davis-Gundy's inequalities* and *Gronwall's lemma* imply that there exists $C_2 = C_2(K_1)$, with $K_1 > 0$ in **Condition 4.2.2**, such that

$$
\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T} |Y_s^\varepsilon - \bar{Y}_s^\varepsilon|^2\Big] + \bar{\mathbb{E}}\Big[\int_t^T |Z_r^\varepsilon - \bar{Z}_r^\varepsilon| dr\Big] + \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} |V_r^\varepsilon - \bar{V}_r^\varepsilon|^2 \nu(dz) dr\Big] \\
&\leq C_2 e^{C_2(T-t)}(T-t)\Big(\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T} |X_s - \bar{X}_s|^2\Big] + \bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T} |Y_s - \bar{Y}_s|^2\Big] \\
&+ \bar{\mathbb{E}}\Big[\int_t^T |Z_s - \bar{Z}_s|^2\Big] + \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} |V_s(z) - \bar{V}_s(z)|^2 \nu(dz) ds\Big]\Big).
\end{aligned}
\tag{4.2.7}
$$

Hence, it follows from (4.2.6) and (4.2.7) that, for every $0 < \varepsilon \leq 1$,

$$
\begin{aligned}
&||\Theta^\varepsilon(X, Y, Z, V) - \Theta^\varepsilon(\bar{X}, \bar{Y}, \bar{Z}, \bar{V})||^2_{\mathcal{M}^2([t,T])} \\
&\leq (T-t)(C_1 + C_2 e^{C_2(T-t)})||(X, Y, Z, V) - (\bar{X}, \bar{Y}, \bar{Z}, \bar{V})||^2_{\mathcal{M}^2[t,T]}.
\end{aligned}
$$

Choosing $\delta > 0$ sufficiently small such that $\delta(C_1 + C_2 e^{C_2\delta}) < 1$, we have that, for every $0 < \varepsilon \leq 1$, Θ^ε is a contraction in the Banach space $\mathcal{M}^2[T-\delta, T]$. Then, for every $T - \delta < T' < T$, Θ^ε has a unique fixed point in $\mathcal{M}^2[T', T]$.

Hence, for every $0 < \varepsilon \leq 1$, for given $t \in [T', T]$, there exists a unique $(X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon, V_s^\varepsilon)_{s\in[t,T]} \in \mathcal{M}^2[t,T]$ that solves (4.2.1) for every $s \in [t,T]$. □

Remark 4.2.4. *For every $0 < \varepsilon \leq 1$, $T' < T$, and $(Y_s^{t,x,\varepsilon})_{s\in[t,T]}$ in the conditions of **Theorem 4.2.1** we introduce the random field*

$$
u^\varepsilon(t, x) := Y_t^{t,x,\varepsilon}, \quad (t, x) \in [T', T] \times \mathbb{R}^d.
$$

*Given $\varepsilon > 0$, it is a well-known fact from the theory of FBSDEs that, since the coefficients of (4.2.1) are deterministic and $u^\varepsilon(t, x)$ is $\tilde{\mathcal{G}}_t$-measurable, the function u^ε is a deterministic function of (t, x). This can be proved by Blumenthal's 0-1 law. The proof follows from analogous arguments of the Brownian case. We refer the reader to Delarue (2002)-**Remark 1.2**.*

Theorem 4.2.2 (**Markovian structure**). *For all $\varepsilon > 0$, $x \in \mathbb{R}^d$, $t \in [T', T]$ with $T' < T$ given in **Theorem 4.2.1** and $\zeta \in L^2(\bar{\mathbb{V}}, \tilde{\mathcal{G}}_t, \bar{\mathbb{P}}, \mathbb{R}^d)$, we have*

$$u^\varepsilon(t, \zeta) = Y_t^{t,x,\zeta}, \quad \bar{\mathbb{P}} - a.s.$$

We refer the reader to *Li and Tang (1999)*.

Remark 4.2.5.

1. *Fix $T > 0$, $t \in [0, T]$ and $0 < \varepsilon, \delta \leq 1$. All the results presented in this section hold for the following FBSDE with jumps*

$$\begin{cases} X_s^{t,x,\varepsilon,\delta} & = x + \displaystyle\int_t^s b(r, X_r^{t,x,\varepsilon,\delta}, Y_r^{t,x,\varepsilon,\delta})dr + \sqrt{\varepsilon} \int_t^s \sigma(r, X_r^{t,x,\varepsilon,\delta}, Y_r^{t,x,\varepsilon,\delta})dB_r \\ & + \delta \displaystyle\int_t^s \int_{\mathbb{R}^d} \beta(X_{r^-}^{t,x,\varepsilon,\delta}, z)\tilde{N}^{\frac{1}{\delta}}(dr, dz), \\ Y_s^{t,x,\varepsilon,\delta} & = g(X_T^{t,x,\varepsilon,\delta}) + \displaystyle\int_s^T f\Big(r, X_r^{t,x,\varepsilon,\delta}, Y_r^{t,x,\varepsilon,\delta}, Z_r^{t,x,\varepsilon,\delta}, \int_{\mathbb{R}^d} V_r^{t,x,\varepsilon,\delta}(z)\frac{\gamma^\delta(z)}{\delta}\nu(dz)\Big)dr \\ & - \displaystyle\int_s^T Z_r^{t,x,\varepsilon,\delta}dB_r - \int_s^T \int_{\mathbb{R}^d} V_{r-}^{t,x,\varepsilon,\delta}(z)\tilde{N}^{\frac{1}{\delta}}(dr, dz), \quad t \leq s \leq T. \end{cases} \tag{4.2.8}$$

2. *As in **Remark 4.2.4**, we define the function*

$$u^{\varepsilon,\delta}(t, x) := Y_t^{t,x,\varepsilon,\delta} \quad (t, x) \in [T', T] \times \mathbb{R}^d,$$

   *for $T' < T$ independent of $\varepsilon, \delta \in (0, 1]$ given by the analogous result of **Theorem 4.2.1** for (4.2.8). When $\varepsilon = \delta$ we write $u^{\varepsilon,\delta}(t, x) = u^\varepsilon(t, x)$, for all $(t, x) \in [T', T] \times \mathbb{R}^d$.*

4.3 Connections with PIDEs

Fix $T > 0$. We impose further the following conditions on the coefficients of (4.2.1).

Condition 4.3.1. We assume that the functions

$$\begin{aligned}
&b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^d,\\
&\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^{d\times d},\\
&\beta : \mathbb{R}^d \longrightarrow \mathbb{R}^d,\\
&f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,\\
&g : \mathbb{R}^d \longrightarrow \mathbb{R}^n
\end{aligned}$$

are smooth with bounded first derivatives.

Theorem 4.3.1. *We assume that **Condition 4.2.2** and **Condition 4.3.1** hold. Then, for every $\varepsilon, \delta > 0$ there exists a unique classical solution $u^{\varepsilon,\delta} \in C^{1,2}([0, T]\times\mathbb{R}^d, \mathbb{R}^n)$ continuously differentiable in the time variable $t \in [0, T]$ and two times continuously differentiable in the space variable $x \in \mathbb{R}^d$ bounded uniformly in*
$\varepsilon, \delta > 0$ of the following terminal value problem

$$\begin{cases} (\partial_t + \mathcal{L}^{\varepsilon,\delta})u^{\varepsilon,\delta}(t, x) \\ +h^\delta\big(t, x, u^{\varepsilon,\delta}(t, x), \sqrt{\varepsilon}\sigma^t(t, x, u^{\varepsilon,\delta}(t, x))\nabla_x u^{\varepsilon,\delta}(t, x), u^{\varepsilon,\delta}(t, x + \delta\beta(x)) - u^{\varepsilon,\delta}(t, x)\big) = 0, \\ u^{\varepsilon,\delta}(T, x) = g(x), \quad x \in \mathbb{R}^d, \quad t \in [0, T], \end{cases} \tag{4.3.1}$$

where, for every $\varepsilon, \delta > 0$ and $(s, x, y, z, k) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^d$,

$$h^\delta(s, x, y, z, k) = f\Big(s, x, y, z, \int_{\mathbb{R}^d} \gamma^\delta(k)\frac{1}{\delta}\nu(\mathrm{d}k)\Big), \tag{4.3.2}$$

the matrix $a := \sigma^T\sigma$ and $\mathcal{L}^{\varepsilon,\delta} = \mathcal{K}_1^\varepsilon + \mathcal{K}_2^\delta$ is the decomposition of this operator into local and nonlocal components with

$$\begin{cases} \mathcal{K}_1^\varepsilon\varphi(t, x) = \langle b(t, x, \varphi(t, x)), \nabla_x\varphi(t, x)\rangle + \frac{\varepsilon}{2}tr(a(t, x, \varphi(t, x))\nabla_x^2\varphi(t, x)), \\ \mathcal{K}_2^\delta\varphi(t, x) = \displaystyle\int_{\mathbb{R}^d} \frac{\varphi(t, x + \delta\beta(x, z)) - \varphi(t, x) - \langle\delta\beta(x, z), \nabla_x\varphi(t, x)\rangle}{\delta}\nu(\mathrm{d}z), \end{cases} \tag{4.3.3}$$

for all $\varphi \in C^{1,2}\big([0, T] \times \mathbb{R}^d, \mathbb{R}^n\big)$.

Moreover, for every $0 < \varepsilon, \delta \leq 1$ for all $t \in [0, T]$ there exists a unique $(\tilde{\mathcal{G}}_s)_{t\leq s\leq T}$-adapted

$$(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, Z_s^{\varepsilon,\delta}, V_s^{\varepsilon,\delta}) \in \mathcal{S}^2(t, T, \mathbb{R}^d) \times \mathcal{S}^2(t, T, \mathbb{R}^n) \times \mathcal{H}^2(t, T, \mathbb{R}^{n\times d}) \times \mathcal{H}_\nu^2(t, T, \mathbb{R}^n)$$

that solves for every $s \in [t, T]$

$$\begin{cases} X_s^{\varepsilon,\delta} & = x + \int_t^s b(r, X_r^{\varepsilon,\delta}, Y_r^{\varepsilon,\delta})dr + \sqrt{\varepsilon}\int_t^s \sigma(r, X_r^{\varepsilon,\delta}, Y_r^{\varepsilon,\delta})dB_r \\ & +\delta \int_t^s \int_{\mathbb{R}^d} \beta(X_{r-}^{\varepsilon,\delta}, z)\tilde{N}^{\frac{1}{\delta}}(dr, dz), \\ Y_s^{\varepsilon,\delta} & = g(X_{T'}^{\varepsilon,\delta}) + \int_s^T f\Big(r, X_r^{\varepsilon,\delta}, Y_r^{\varepsilon,\delta}, Z_r^{\varepsilon,\delta}, \int_{\mathbb{R}^d} V_r^{\varepsilon,\delta}(z)\gamma^\delta(z)\frac{1}{\delta}\nu(dz)\Big)dr \\ & - \int_s^T Z_r^{\varepsilon,\delta} dB_r - \int_s^T \int_{\mathbb{R}^d} V_r^{\varepsilon,\delta}(z)\tilde{N}^{\frac{1}{\delta}}(dr, dz). \end{cases} \tag{4.3.4}$$

Futhermore, we have the following representation formulas,

$$\begin{aligned} Y_s^{\varepsilon,\delta} &= u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta}) \\ Z_s^{\varepsilon,\delta} &= \sqrt{\varepsilon}\nabla_x u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta})\sigma(s, X_s^{\varepsilon,\delta}, u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta})) \\ V_s^{\varepsilon,\delta} &= u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta} + \delta\beta(X_s^{\varepsilon,\delta}, z)) - u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta}), \\ & \quad \text{for all } t \le s \le T, z \in \mathbb{R}^d. \end{aligned} \tag{4.3.5}$$

Remark 4.3.2.

*i) This result follows from Ma et al (2010)-**Theorem 2**. **Theorem 4.3.1** establishes a probabilistic representation of the classical solution of the PIDE (4.3.1), showing that we can find the unique solution of the FBSDE (4.3.4) which has the representation given by (4.3.5). In the literature this representation is called nonlinear Feynman-Kac's formula.*

ii) If $b(t, x, y) = y$, $\sigma = Id$ and $\nu(dz) = \frac{1}{|z|^{d+\alpha}}dz$ for some $\alpha \in (0, 2)$, for every $\varepsilon, \delta > 0$, the corresponding terminal value problem (4.3.1) reads as the terminal value problem for the backwards fractal Burgers equation with a diffusive term

$$\partial_t u + \langle \nabla_x u, u\rangle + \frac{\varepsilon}{2}\Delta u + \delta(-\Delta)^{\frac{\alpha}{2}} u + h^\delta = 0, \tag{4.3.6}$$

that was discussed in the introduction, with h^δ defined in (4.3.2).
The asymptotic study of $\varepsilon, \delta \to 0$ in the associated system of FBSDEs is the probabilistic counterpart for the study of vanishing local viscosity $\varepsilon \to 0$ and nonlocal viscosity $\delta \to 0$ in the fractal Burgers equations.

Nevertheless, the preceding result relies on the smoothness of the function $u^{\varepsilon,\delta}$. If we require less regularity on the function $u^{\varepsilon,\delta}$, there is still a link between the FBSDE system (4.3.4) and the PIDE (4.3.1), via the notion of viscosity solution that we present below.
For every $\delta > 0$, we define the following operator,

$$\mathcal{J}^\delta \varphi(t, x) = \int_{\mathbb{R}^d} \frac{\varphi(t, x + \delta\beta(x, z)) - \varphi(t, x)}{\delta}\gamma^\delta(z)\nu(dz), \text{ for all } \varphi \in C([0, T] \times \mathbb{R}^d, \mathbb{R}^n).$$

We now define the notion of viscosity solution for the terminal value problem (4.3.1). It is a central concept that links FBSDEs and optimal control. BSDEs are equations that characterize the adjoint equation for a stochastic optimal control problem, via the dynamic programming principle. They form an alternative to the Hamilton-Jacobi-Bellman equation that is associated to the control problem. In sophisticated models, we cannot guarantee differentiability or more regularity of the value function that solves the PIDE. Nevertheless, we can associate to the backward process a notion of solution that encodes a generalization of a certain maximum principle for the PIDE. We refer the reader to *Delong* (2013) for more information about BSDEs with jumps and viscosity solutions and *Li and Wei* (2015) for connections between optimal control, viscosity solutions and coupled FBSDEs with jumps.

Definition 4.3.1 (**Viscosity solution of (4.3.1)**).

1. *For every $\varepsilon, \delta > 0$, a continuous function $u^{\varepsilon,\delta} \in C([0,T] \times \mathbb{R}^d)$ is a viscosity subsolution of the terminal value problem (4.3.1) if, for all $i \in \{1,...,n\}$, $u_i^{\varepsilon,\delta}(T,x) \leq g(x)$ for all $x \in \mathbb{R}^d$ and if for any $1 \leq i \leq n$ and for every $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$, whenever $(t,x) \in [0,T] \times \mathbb{R}^d$ is a strict local point of maximum of $u_i^{\varepsilon,\delta} - \varphi$ we have*

$$
\begin{aligned}
&- \partial_t \varphi(t,x) - \mathcal{L}^{\varepsilon,\delta}\varphi(t,x) \\
&- f(t,x,u^{\varepsilon,\delta}(t,x), \sqrt{\varepsilon}\nabla_x\varphi(t,x)\sigma(t,x,u^{\varepsilon,\delta}(t,x)), \mathcal{J}^\delta\varphi(t,x)) \leq 0.
\end{aligned}
$$

2. *A function $u^{\varepsilon,\delta} \in C([0,T] \times \mathbb{R}^d)$ is a viscosity supersolution of the terminal value problem (4.3.1) if, for all $i \in \{1,...,n\}$, $u_i^{\varepsilon,\delta}(T,x) \geq g(x)$ for all $x \in \mathbb{R}^d$ and if for any $1 \leq i \leq n$, for every $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$, whenever $(t,x) \in [0,T] \times \mathbb{R}^d$ is a strict local point of minimum of $u_i^{\varepsilon,\delta} - \varphi$ we have*

$$
\begin{aligned}
&- \partial_t \varphi(t,x) - \mathcal{L}^{\varepsilon,\delta}\varphi(t,x) \\
&- f(t,x,u^{\varepsilon,\delta}(t,x), \sqrt{\varepsilon}\nabla_x\varphi(t,x)\sigma(t,x,u^{\varepsilon,\delta}(t,x)), \mathcal{J}^\delta\varphi(t,x)) \geq 0.
\end{aligned}
$$

3. *A function $u^{\varepsilon,\delta} \in C([0,T] \times \mathbb{R}^d)$ of (4.3.1) is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution of (4.3.1).*

The following theorem states that even if the coefficients of the FBSDE system (4.3.4) are not differentiable we have a nonlinear Feynman-Kac formula holding, in the sense of viscosity solutions.
For connections between the notion of weak solution in the Sobolev sense for terminal value problems such as (4.3.1) and FBSDE with jumps we refer to *Matoussi and Wang (2009)* and further references that can be found there.

In what follows we always assume $T' < T$ in the conditions of **Theorem 4.2.1** and $t \in [T', T]$.

Theorem 4.3.2. *For every $0 \leq \varepsilon, \delta \leq 1$, under **Condition 4.2.2**, the function $u^{\varepsilon,\delta}(t,x) := Y_t^{\varepsilon,\delta}$, where $(Y_s^{\varepsilon,\delta})_{t\leq s\leq T}$ is the solution of the backward SDE of (4.3.4), in the sense of **Theorem 4.2.1**, is a viscosity solution of the terminal value problem (4.3.1).*

We state a collection of estimates that will be useful in the sequel of this chapter. For every $t \in [T', T]$ let $(X_s^{t,x,\varepsilon}, Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}, V_s^{t,x,\varepsilon})_{t\leq s\leq T} \in \mathcal{M}^2[t,T]$ be the unique solution of (4.2.1) for every $s \in [t,T]$.

Proposition 4.3.1. *Fix $p \geq 2$ and $\rho \geq 0$, $t' \in [T', T]$, $t' > t$ and $x' \in \mathbb{R}^d$. We have the following energy estimates for the state process $(X_s^{t,x,\varepsilon})_{t\leq s\leq T}$ and for the backward process $(Y_s^{t,x,\varepsilon})_{t\leq s\leq T}$: there exists $K > 0$ depending on the constants $K_1, K_2, K_3 > 0$ of **Condition 4.2.2** and independent of $\varepsilon > 0$ such that*

$$\begin{aligned}
\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T} |X_s^{t,x,\varepsilon}|^p\Big] &\leq K(1+|x|^p)(T-t),\\
\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T} |X_s^{t,x,\varepsilon} - X_s^{t,x',\varepsilon}|^p\Big] &\leq K|x-x'|^p,\\
\bar{\mathbb{E}}\Big[\sup_{t'\leq s\leq T} |X_s^{t,x,\varepsilon} - X_s^{t',x',\varepsilon}|^2\Big] &\leq K(|x-x'|^2 + (1+|x|^2 \vee |x'|^2)|t-t'|),
\end{aligned} \tag{4.3.7}$$

and

$$\begin{aligned}
&||Y^{t,x,\varepsilon}||^2_{\rho,\mathcal{S}^2([t,T],\mathbb{R}^k)}\\
&\leq K\Big(\bar{\mathbb{E}}[e^{\rho T}|g(X_T^{t,x,\varepsilon})|^2]\\
&+ \bar{\mathbb{E}}\Big[\int_t^T e^{\rho s}\Big|f\Big(s, X_s^{\varepsilon,t,x}, Y_s^{\varepsilon,t,x}, Z_s^{\varepsilon,t,x}, \int_{\mathbb{R}^d} V_s^{\varepsilon,t,x}(z)\gamma^\varepsilon(z)\frac{1}{\varepsilon}\nu(dz)\Big)\Big|^2 ds\Big]\Big).
\end{aligned} \tag{4.3.8}$$

The estimates (4.3.7) follow from **Proposition 3.1** in *Li and Wei (2014)* and (4.3.8) is proven in *Delong (2013)* - **Lemma 3.1.1**.

From the previous proposition we conclude that the function u^ε from **Theorem 4.2.2** is Lipschitz continuous in the space variable $x \in \mathbb{R}^d$ and locally $\frac{1}{2}$- Hölder continuous in the time variable $t \in [0,T]$.

Proposition 4.3.2. *For every $0 < \varepsilon \leq 1$, under **Condition 4.2.2** the function u^ε from **Theorem 4.2.2** is continuous on $[0,T] \times \mathbb{R}^d$. Moreover, u^ε satisfies*

$$\begin{aligned}
&|u^\varepsilon(t,x) - u^\varepsilon(t',x')|^2\\
&\leq K(|x-x'|^2 + (1+|x|^2 \vee |x'|^2)|t-t'|), \quad \text{for all } (t,x),(t',x') \in [T',T]\times\mathbb{R}^d,
\end{aligned} \tag{4.3.9}$$

*for some $K = K(K_1, K_2, K_3, T, T')$, where $K_1, K_2, K_3 > 0$ are constants from **Condition 4.2.2**.*

Proof. Consider the following BSDE

$$
\begin{aligned}
Y_s^{t,x,\varepsilon} =& g(X_T^{t,x,\varepsilon}) \\
&+ \int_s^T \mathbf{1}_{[t,T]}(r) f\Big(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon}, Z_r^{t,x,\varepsilon}, \int_{\mathbb{R}^d} V_r^{t,x,\varepsilon}(z)\gamma^\varepsilon(z)\frac{1}{\varepsilon}\nu(dz)\Big) dr \\
&- \int_s^T Z_r^{t,x,\varepsilon} dB_r - \int_s^T \int_{\mathbb{R}^d} V_{r-}^{t,x,\varepsilon}(z)\tilde{N}^{\frac{1}{\varepsilon}}(dr, dz), \quad t \le s \le T.
\end{aligned}
$$

Let us define $X_r^{t,x,\varepsilon} = x$, $Y_r^{t,x,\varepsilon} = Y_t^{t,x,\varepsilon} = u^\varepsilon(t,x)$ and $Z_r^{t,x,\varepsilon} = V_r^{t,x,\varepsilon} = 0$ for $T' < r \le t$. Using the estimate (4.3.8) ($\rho = 0$), we derive that,
for some constant $K = K(K_1, K_2, K_3, T, T')$ that may change from line to line, we have for all $(t,x) \in [T', T] \times \mathbb{R}^d$

$$
\begin{aligned}
|u^\varepsilon(t,x)|^2 &= |Y_t^{t,x,\varepsilon}|^2 \\
&\le ||Y^{t,x,\varepsilon}||_{S^2(t,T,\mathbb{R}^n)} \\
&\le K\bar{\mathbb{E}}\Big[|g(X_T^{t,x,\varepsilon})|^2 + \int_0^T \mathbf{1}_{[t,T]}(r)|f(r, X_r^{t,x,\varepsilon}, u^\varepsilon(t,x), 0, 0)|dr\Big] \\
&\le K\Big(\bar{\mathbb{E}}\Big[1 + \sup_{0\le r\le T}|X_r^{\varepsilon,t,x}|^2\Big] + \int_t^T |u^\varepsilon(r,x)|dr\Big) \\
&\le K_1(1 + |x|^2).
\end{aligned} \tag{4.3.10}
$$

In the last inequality we used the *backward Gronwall's inequality* (**Proposition A.1.2**). Let $t' \ge t$. For $T' < r \le t'$ we define $X_r^{t,x,\varepsilon} = x$, $Y_r^{t,x,\varepsilon} = Y_{t'}^{t,x,\varepsilon} = u^\varepsilon(t', x)$ and $Z_r^{t,x,\varepsilon} = V_r^{t,x,\varepsilon} = 0$. Due to the estimate (4.3.8), the Lipschitz assumptions on γ^ε and f made in **Condition 4.2.2** it follows, for some $K = K(K_1, K_2, K_3, T, T')$, that may differ from line to line eventually,

$$
\begin{aligned}
|Y_t^{t,x,\varepsilon} - Y_{t'}^{t',x',\varepsilon}|^2 &= |Y_0^{t,x,\varepsilon} - Y_0^{t',x',\varepsilon}|^2 \\
&\leq \bar{\mathbb{E}}\Big[\sup_{0\leq s\leq T} |Y_s^{t,x,\varepsilon} - Y_s^{t',x',\varepsilon}|^2\Big] \\
&\leq K\mathbb{E}\Big[|g(X_T^{t,x,\varepsilon}) - g(X_T^{t',x',\varepsilon})|^2 + \\
&\int_0^T \Big|\mathbf{1}_{[t,T]}(r) f\Big(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon}, Z_r^{t,x,\varepsilon}, \int_{\mathbb{R}^d} V_r^{t,x,\varepsilon}(z)\gamma^\varepsilon(z)\frac{1}{\varepsilon}\nu(dz)\Big) \\
&- \mathbf{1}_{[t',T]}(r) f\Big(r, X_r^{t',x',\varepsilon}, Y_r^{t',x',\varepsilon}, Z_r^{t',x',\varepsilon}, \int_{\mathbb{R}^d} V_r^{t',x',\varepsilon}(z)\gamma^\varepsilon(z)\frac{1}{\varepsilon}\nu(dz)\Big)\Big|^2 dr\Big] \\
&\leq K\bar{\mathbb{E}}\Big[|X_T^{t,x,\varepsilon} - X_T^{t',x',\varepsilon}|^2 + \int_{t'}^T |X_r^{t,x,\varepsilon} - X_r^{t',x',\varepsilon}|^2 dr \\
&+ \int_t^{t'} |f\Big(r, X_r^{t,x,\varepsilon}, Y_r^{t',x',\varepsilon}, Z_r^{t',x',\varepsilon}, \int_{\mathbb{R}^d} V_r^{t',x',\varepsilon}(z)\gamma^\varepsilon(z)\frac{1}{\varepsilon}\nu(dz)\Big)\Big|^2 dr\Big] \\
&\leq K\bar{\mathbb{E}}\Big[\sup_{0\leq r\leq T} |X_r^{t,x,\varepsilon} - X_r^{t',x',\varepsilon}|^2 \\
&+ \int_t^{t'} \Big(1 + |X_r^{t,x,\varepsilon}|^2 + |Y_r^{t',x',\varepsilon}|^2 \\
&+ |Z_r^{t',x',\varepsilon}|^2 + \int_{\mathbb{R}^d} |V_r^{t',x',\varepsilon}|^2 \frac{1}{\varepsilon}\nu(dz)\Big) dr\Big].
\end{aligned}
$$

Since $Y_r^{t',x',\varepsilon} = u^\varepsilon(t',x')$ and $Z_r^{t',x',\varepsilon} = V_r^{t',x',\varepsilon} = 0$ for $r \leq t'$, (4.3.10) and (4.3.7) yield, for some $K > 0$, that may differ from line to line,

$$
\begin{aligned}
|Y_t^{t,x,\varepsilon} - Y_{t'}^{t',x',\varepsilon}|^2 &\leq K\bar{\mathbb{E}}\Big[\sup_{0\leq r\leq T} |X_r^{t,x,\varepsilon} - X_r^{t',x',\varepsilon}|^2 \\
&+ \int_t^{t'} \Big(1 + |x|^2 + |X_r^{t,x,\varepsilon}|^2 + |u^\varepsilon(t',x')|^2\Big) dr\Big] \\
&\leq K\Big(|x - x'|^2 + (1 + |x|^2 \vee |x'|^2)|t - t'|\Big).
\end{aligned}
$$

This finishes the proof. □

Remark 4.3.3. *For fixed $\varepsilon, \delta \in (0,1]$ the same conclusion follows for $u^{\varepsilon,\delta}$ defined in the second statement of **Remark 4.2.5**, i.e. there exists $K = K(K_1, K_2, T, T')$ independent of ε and δ such that*

$$
\begin{aligned}
&|u^{\varepsilon,\delta}(t,x) - u^{\varepsilon,\delta}(t',x')| \\
&\leq K(|x - x'|^2 + (1 + |x|^2 \vee |x'|^2)|t - t'|), \quad (t,x), (t',x') \in [T',T] \times \mathbb{R}^d. \qquad (4.3.11)
\end{aligned}
$$

The following proposition, known as comparison theorem, will be useful in the sequel to prove the nonlinear *Feynman-Kac formula*, via the link of viscosity solutions for the terminal value problem (4.3.1) and the backward process of (4.3.4).

Proposition 4.3.3 (**Comparison principle**). *Assume $n = 1$ and consider $(b, \sigma, f^j, g^j)_{j=1,2}$ under* ***Condition 4.2.2****. Let us fix $\varepsilon, \delta \in (0, 1]$ and $x \in \mathbb{R}^d$.*
Due to ***Theorem 4.2.1****, let $T^{(1)} < T$, some arbitrarily fixed $t \in [T^{(1)}, T]$ and $(X_s^{\varepsilon,\delta,j}, Y_s^{\varepsilon,\delta,j}, Z_s^{\varepsilon,\delta,j}, V_s^{\varepsilon,\delta,j})_{t \le s \le T}$ be the respective solutions of (4.3.4) with coefficients (b, σ, f^j, g^j) on the time interval $[t, T]$.*
If for every $\delta > 0$ and $(s, x, y, z, k) \in [t, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^d$

$$f^1\Big(s, x, y, z, \int_{\mathbb{R}^d} k(z)\gamma^\delta(z)\frac{1}{\delta}\nu(dz)\Big) \ge f^2\Big(s, x, y, z, \int_{\mathbb{R}^d} k(z)\gamma^\delta(z)\frac{1}{\delta}\nu(dz)\Big)$$

and

$$g^1(x) \ge g^2(x) \text{ for all } x \in \mathbb{R}^d,$$

then

$$Y_s^{\varepsilon,\delta,1} \ge Y_s^{\varepsilon,\delta,2}, \quad s \in [t, T].$$

We refer the reader to *Wu (2003)* for a proof.

Proof of Theorem 4.3.2

Proof. The proof follows the arguments of *Barles et al.(1996)*-**Theorem 3.4**. In our case, the FBSDE (4.3.4) is coupled but the reasoning is similar. For this reason, we only sketch a proof.
We remark that in the definition of viscosity solution for (4.3.1) we ask the properties of strict local maximum (resp. strict local minimum) and not the global maximum (resp. globa minumum) that is required in the definition of viscosity solution presented in *Barles et al. (1996)*.
Fix $\varepsilon, \delta \in (0, 1]$ and $x \in \mathbb{R}^d$. Using **Theorem 4.2.1** and **Theorem 4.2.2**, there exists $T' < T$ independent of $\varepsilon, \delta > 0$ such that for every fixed $t \in [T', T]$ we have a unique solution of (4.3.4)

$$(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, Z_s^{\varepsilon,\delta}, V_s^{\varepsilon,\delta}) \in \mathcal{M}^2[t, T]$$

and a measurable function

$$u^{\varepsilon,\delta} : [T', T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^n$$

such that

$$Y_s^{\varepsilon,\delta} = u^{\varepsilon,\delta}(s, X_s^{\varepsilon,\delta}), \quad \text{for all } t \le s \le T.$$

We prove that $u_i^{\varepsilon,\delta}$ is a viscosity subsolution of (4.3.1). A similar argument shows that $u^{\varepsilon,\delta}$ is also a viscosity supersolution of (4.3.1).
Due to **Proposition 4.3.2**, $u^{\varepsilon,\delta} \in C([T', T] \times \mathbb{R}^d)$.
We fix $(t, x) \in [T', T] \times \mathbb{R}^d$, $i \in \{1, \dots, n\}$ and a function $\varphi \in C^{1,2}([T', T] \times \mathbb{R}^d)$ such that (t, x) is a strict local point of maximum of $u^{\varepsilon,\delta} - \varphi$. We assume without loss of generality that $u_i^{\varepsilon,\delta}(t, x) = \varphi(t, x)$ and $u_i^{\varepsilon,\delta} < \varphi$ in some neighborhood of (t, x). We can consider $\varphi \in C_b^\infty([T', T] \times \mathbb{R}^d)$ due to a standard approximation argument: we can find a sequence

of continuously differentiable functions with bounded derivatives $\varphi_n \in C_b^\infty([T',T]\times\mathbb{R}^d)$ such that $(\varphi_n)_{n\in\mathbb{N}}$, and first and second derivatives respectively, converge to φ, and first and second derivatives of φ respectively, in the uniform topology on compact sets of $[T',T]\times\mathbb{R}^d$, prove the subsolution property for $(\varphi_n)_{n\in\mathbb{N}}$ and then pass to the limit. For details we refer *El Karoui et al. (1997)*-**Theorem 4.2**.

1. We prove the viscosity subsolution by contradiction. Firts, we derive some estimates that will be useful for the sequel. We choose $h>0$ such that $t+h\le T$ and such that $u_i^{\varepsilon,\delta}(s,y)<\varphi(s,y)$ for $s\in[t,t+h]$ and $y\in B_h(x)$.
   We define the stopping time
   $$\xi := \inf\{s\ge t \mid |X_s^{\varepsilon,\delta}-x|\ge h\}\wedge(t+h).$$
   The process $(Y_s^{\varepsilon,\delta})_{t\le s\le T}$ satisfies, due to **Theorem 4.2.2**,
   $$\begin{aligned} Y_s^{\varepsilon,\delta} &= u(t+h,X_{t+h}^{\varepsilon,\delta})\\ &+\int_s^{t+h} f\Big(r,X_r^{\varepsilon,\delta},Y_r^{\varepsilon\delta},Z_r^{\varepsilon,\delta},\int_{\mathbb{R}^d}V_r^{\varepsilon,\delta}(z)\frac{1}{\delta}\nu(dz)\Big)dr\\ &-\int_s^{t+h} Z_r^{\varepsilon,\delta}dB_r-\int_s^{t+h}\int_{\mathbb{R}^d}V_r^{\varepsilon,\delta}(z)\tilde{N}^{\frac{1}{\delta}}(dr,dz),\quad t\le s\le t+h. \end{aligned}\tag{4.3.12}$$
   Given $y\in\mathbb{R}$ and $z\in\mathbb{R}^k$, the vector $(y,\tilde{z}_i)$ denotes the $k-$dimensional vector whose i-th component equals to y and all other components equal the corresponding ones of z. We consider the following one dimensional BSDE:
   $$\begin{aligned} \bar{Y}_i^{\varepsilon,\delta}(s) =&\varphi(t+h,\bar{X}_{t+h}^{\varepsilon,\delta})\\ &+\int_s^{t+h}\mathbf{1}_{[0,\xi]}(r)f_i\Big(r,X_r^{\varepsilon,\delta},(\bar{Y}_i^{\varepsilon,\delta}(r),\tilde{Y}_i^{\varepsilon\delta}(r)),\bar{Z}_r^{\varepsilon,\delta},\int_{\mathbb{R}^d}\bar{V}_r^{\varepsilon,\delta}(z)\frac{1}{\delta}\nu(dz)\Big)dr\\ &-\int_s^{t+h}\bar{Z}_r^{\varepsilon,\delta}dB_r-\int_s^{t+h}\int_{\mathbb{R}^d}\bar{V}_r^{\varepsilon,\delta}(z)\tilde{N}^{\frac{1}{\delta}}(drdz),\quad t\le s\le t+h. \end{aligned}\tag{4.3.13}$$
   Since $\varphi\in C_b^\infty([T',T]\times\mathbb{R}^d)$, and in particular it is a Lipschitz bounded function, due to **Theorem 4.2.2**, there exists $T''<T$ (let us write for sake of simplicity of notation $T''=T'$) and a unique solution of (4.3.13)
   $$(\bar{X}_s^{\varepsilon,\delta},\bar{Y}_i^{\varepsilon,\delta}(s),\bar{Z}_s^{\varepsilon,\delta},\bar{V}_s^{\varepsilon,\delta})\in\mathcal{M}^2[T'T].$$
   Since we have $\varphi(s,y)\ge u_i^{\varepsilon,\delta}(s,y)$, for $s\in[t,t+h]$ and $|y-x|<h$, from the comparison principle stated in **Proposition 4.3.3** it follows that
   $$\bar{Y}_i^{\varepsilon,\delta}(s)\ge Y_i^{\varepsilon,\delta}(s),\quad t\le s\le t+h.\tag{4.3.14}$$
   In particular, $\bar{Y}_i^{\varepsilon,\delta}(t)\ge u_i^{\varepsilon,\delta}(t,x)=\varphi(t,x)$.
   We define further, for all $\varepsilon,\delta>0$,
   $$\begin{aligned} &\Theta^{\varepsilon,\delta}(s,x) := \mathbf{1}_{[0,\xi]}(t)(\partial_t\varphi(t,x)+\mathcal{L}^{\varepsilon,\delta}\varphi(t,x)),\quad (s,x)\in[T',T]\times\mathbb{R}^d,\\ &\Gamma^\delta(s,x,z) := \mathbf{1}_{[0,\xi]}(t)(\varphi(t,x+\delta\beta(x,z))-\varphi(t,x)),\quad (s,x,z)\in[T',T]\times\mathbb{R}^d\times\mathbb{R}^d. \end{aligned}$$

Since $\varphi \in C_b^\infty([T',T]\times\mathbb{R})$, there exists some constant $K>0$ such that

$$
\begin{aligned}
&|\Theta^{\varepsilon,\delta}(s,x)| \le K(1+|x|^2),\\
&|\Gamma^\delta(s,x,z)| \le K(1\wedge|z|), \quad (s,x,z)\in[T',T]\times\mathbb{R}^d\times\mathbb{R}^d.
\end{aligned}
$$

Define, for $t\le s\le t+h$ and $\varepsilon,\delta>0$,

$$
\begin{aligned}
&\hat{Y}_i^{\varepsilon,\delta}(s) = \bar{Y}_i^{\varepsilon,\delta}(s\wedge\xi) - \varphi(s, X_{s\wedge\xi}^{\varepsilon,\delta}),\\
&\hat{Z}_i^{\varepsilon,\delta}(s) = \mathbf{1}_{[0,\xi]}(s)(\bar{Z}_i^{\varepsilon,\delta}(s) - \sqrt{\varepsilon}\nabla\varphi(s,X_s^{\varepsilon,\delta})\sigma(s,X_s^{\varepsilon,\delta},Y_s^{\varepsilon,\delta})) \text{ and}\\
&\hat{V}_i^{\varepsilon,\delta}(s) = \mathbf{1}_{[0,\xi]}(s)(\bar{V}_i^{\varepsilon,\delta}(s) - \Gamma(s,X^{\varepsilon,\delta}(s^-),z)).
\end{aligned}
$$

Itô's formula (**Proposition B.3.1**) yields

$$
\begin{aligned}
\varphi(s,X_s^{\varepsilon,\delta}) = \varphi(t+h, X_{t+h}^{\varepsilon,\delta}) &- \int_s^{t+h}\Theta^{\varepsilon,\delta}(r,X_r^{\varepsilon,\delta})dr\\
&- \sqrt{\varepsilon}\int_s^{t+h}\nabla_x\varphi(r,X_r^{\varepsilon,\delta})\sigma(r,X_r^{\varepsilon,\delta},Y_r^{\varepsilon,\delta})dB_r\\
&- \int_s^{t+h}\int_{\mathbb{R}^d}\Gamma^\delta(r,X_r^{\varepsilon,\delta},z)\tilde{N}^{\frac{1}{\delta}}(dr,dz), \quad t\le s\le t+h.
\end{aligned}
\tag{4.3.15}
$$

From (4.3.13) and (4.3.15), it follows that $(\hat{Y}_i^{\varepsilon,\delta}(s), \hat{Z}_i^{\varepsilon,\delta}(s), \hat{V}_i^{\varepsilon,\delta}(s))_{t\le s\le t+h}$ is the unique solution of

$$
\begin{aligned}
\hat{Y}_i^{\varepsilon,\delta}(s) = \int_s^{t+h} &\Theta^{\varepsilon,\delta}(r,X_r^{\varepsilon,\delta})\\
&+ f_i\Big(r, X_r^{\varepsilon,\delta}, \varphi(r,X_r^{\varepsilon,\delta}) + \hat{Y}_i^{\varepsilon,\delta}(s),\\
&\sqrt{\varepsilon}\nabla\varphi(s,X_s^{\varepsilon,\delta})\sigma(s,X_s^{\varepsilon,\delta},Y_s^{\varepsilon,\delta}) + \hat{Z}_i^{\varepsilon,\delta}(r), \int_{\mathcal{X}}(\Gamma(r,X_{r^-}^{\varepsilon,\delta},z) + \hat{V}_i^{\varepsilon,\delta}(s))\frac{\gamma^\delta(z)}{\delta}\nu(dz)\Big)dr\\
&- \int_s^T\hat{Z}_i^{\varepsilon,\delta}(r)dB_r - \int_s^T\int_{\mathbb{R}^d}\hat{V}_i^{\varepsilon,\delta}(r)\tilde{N}^{\frac{1}{\delta}}(dr,dz), \quad t\le s\le t+h.
\end{aligned}
\tag{4.3.16}
$$

As in the proof of *Delong (2013)*-**Proposition 3.1.2**, using *Itô's formula* (**Proposition B.3.2**) and the inequalities of *Burkholder-Davis-Gundy* (**Proposition B.3.3**), due to the growth assumptions made on $f,\sigma,\Theta^{\varepsilon,\delta}$ and Γ^δ, we can derive, for some $K=K(K_1,K_2,K_3,T,T')>0$, where $K_1,K_2,K_3>0$ are constants given in

Condition 4.2.2, that differ from line to line,

$$
\begin{aligned}
&\bar{\mathbb{E}}[|\hat{Y}_i^{\varepsilon,\delta}(s)|^2] + \mathbb{E}\Big[\int_s^{t+h} |\hat{Z}_i^{\varepsilon,\delta}(r)|^2 dr\Big] + \bar{\mathbb{E}}\Big[\int_s^{t+h}\int_{\mathbb{R}^d} \frac{1}{\delta}|\hat{V}_i^{\varepsilon,\delta}(r,z)|\nu(dz)dr\Big] \\
&\leq K\mathbb{E}\Big[\int_s^{t+h} |\hat{Y}_i^{\varepsilon,\delta}(r)|\Big|\Theta(r, X_r^{\varepsilon,\delta}) \\
&+ f_i\Big(r, X_r^{\varepsilon,\delta}, \sqrt{\varepsilon}\nabla_x\varphi(t,x)\sigma(r, X_r^{\varepsilon,\delta}, Y_r^{\varepsilon,\delta}), \int_{\mathbb{R}^d}\Gamma^\delta(r, X_r^{\varepsilon,\delta}, z)\frac{\gamma^\delta(z)}{\delta}\nu(dz)\Big)\Big|\Big] \\
&\leq K\bar{\mathbb{E}}\Big[\int_s^{t+h} |\hat{Y}_i^{\varepsilon,\delta}(r)|(1+|X_r^{\varepsilon,\delta}|^2)dr\Big] \\
&\leq K\bar{\mathbb{E}}\Big[\int_s^{t+h} |\hat{Y}_i^{\varepsilon,\delta}(r)|(1+|x|^2+|X_r^{\varepsilon,\delta}-x|^2)dr\Big] \\
&\leq K\bar{\mathbb{E}}\Big[\int_s^{t+h} (|\hat{Y}_i^{\varepsilon,\delta}(r)| + |\hat{Y}_i^{\varepsilon,\delta}(r)|^2 + |X_r^{\varepsilon,\delta}-x|^4)dr\Big], \quad t \leq s \leq t+h.
\end{aligned}
$$

The trivial inequality $|y| \leq 1 + |y|^2$ and (4.3.7) yields, for some $K > 0$,

$$
\begin{aligned}
\bar{\mathbb{E}}[|\tilde{Y}_i^{\varepsilon,\delta}(r)|^2] &\leq K\bar{\mathbb{E}}\Big[\int_s^{t+h} (1+|\tilde{Y}_i^{\varepsilon,\delta}(r)|^2 + h)dr\Big] \\
&\leq K\Big(h + h^2 + \bar{\mathbb{E}}\Big[\int_t^{t+h} |\tilde{Y}_i^{\varepsilon,\delta}(r)|^2 dr\Big]\Big), \quad t \leq s \leq t+h.
\end{aligned}
$$

Using *Gronwall's inequality*, we have, for some $K > 0$,

$$
\bar{\mathbb{E}}[|\tilde{Y}_i^{\varepsilon,\delta}(r)|^2] \leq K(h+h^2)e^{Kh} \leq Kh, \quad t \leq s \leq t+h,
$$

and therefore,

$$
\bar{\mathbb{E}}[|\tilde{Y}_i^{\varepsilon,\delta}(r)|^2] \leq K\sqrt{h}, \quad t \leq s \leq t+h. \tag{4.3.17}
$$

For $h > 0$ sufficiently small, (4.3.16) and (4.3.17) imply, for some $K > 0$,

$$
\begin{aligned}
&\bar{\mathbb{E}}\Big[\int_t^{t+h} |\hat{Z}_i^{\varepsilon,\delta}(r)|^2 dr + \int_t^{t+h}\int_{\mathbb{R}^d} |\hat{V}_i^{\varepsilon,\delta}(r,z)|^2 \frac{\nu(dz)}{\delta}dr\Big] \\
&\leq K\int_t^{t+h} (\sqrt{h} + h + h)dr \\
&\leq K\sqrt{h}.
\end{aligned} \tag{4.3.18}
$$

2. We prove the subsolution property by contradiction. Suppose that for the given $(t,x) \in [T', T] \times \mathbb{R}^d$, such that (t,x) is a strict local point of maximum of $u_i^{\varepsilon,\delta} - \varphi$ we have

$$
\begin{aligned}
&- \partial_t\varphi(t,x) - \mathcal{L}^{\varepsilon,\delta}\varphi(t,x) \\
&- f(t, x, u^{\varepsilon,\delta}(t,x), \sqrt{\varepsilon}\nabla_x\varphi(t,x)\sigma(t,x,u^{\varepsilon,\delta}(t,x)), \mathcal{J}^\delta\varphi(t,x)) > 0.
\end{aligned} \tag{4.3.19}
$$

Let therefore $\tau > 0$ and $h_0, h_0' > 0$ such that for $T < t - h_0 \le r \le t + h_0$ and for $|y - x| \le h_0'$, $\varphi(r, y) > u^{\varepsilon,\delta}(r, y)$ and

$$\begin{aligned}
&- \partial_t \varphi(r, y) - \mathcal{L}^{\varepsilon,\delta} \varphi(r, y) \\
&- f(r, y, u^{\varepsilon,\delta}(r, y), \sqrt{\varepsilon} \nabla_x \varphi(r, y) \sigma(r, y, u^{\varepsilon,\delta}(r, y)), \mathcal{J}^\delta \varphi(r, y)) \ge \tau.
\end{aligned} \tag{4.3.20}$$

Given $\varepsilon, \delta > 0$, we choose $h > 0$ small enough such that $h < h_0$ and we define

$$\xi_h := \frac{1}{h} \bar{\mathbb{E}}\Big[\int_t^{t+h} U^{\varepsilon,\delta}(r, X_r^{\varepsilon,\delta}) dr \Big], \tag{4.3.21}$$

where

$$\begin{aligned}
U^{\varepsilon,\delta}(r, y) &:= \partial_t \varphi + \mathcal{L}^{\varepsilon,\delta} \varphi(r, y) \\
&\quad + f(r, y, \varphi(r, y), \sqrt{\varepsilon} \nabla_x \varphi(r, y) \sigma(t, x, u^{\varepsilon,\delta}(r, y)), \mathcal{J}^\delta \varphi(r, y)).
\end{aligned} \tag{4.3.22}$$

Clearly due to the sublinear growth of f and $\varphi \in C_b^\infty([T', T] \times \mathbb{R}^d)$, for some $K > 0$ we have

$$|U^{\varepsilon,\delta}(r, y)| \le K(1 + |y|^2), \quad (r, y) \in [T', T] \times \mathbb{R}^d.$$

Define the following stopping time

$$\zeta := \inf\{t \ge T' \mid |X_s^{\varepsilon,\delta} - x| > h_0'\}.$$

Hence, the a-priori estimate (4.3.7) and *Chebyshev's inequality* yield

$$\begin{aligned}
\bar{\mathbb{P}}(\zeta \le h) &= \bar{\mathbb{P}}\Big(\sup_{t \le s \le t+h} |X_s^{\varepsilon,\delta} - x| > h_0' \Big) \\
&\le \frac{1}{|h_0'|^2} \bar{\mathbb{E}}\Big[\sup_{t \le s \le t+h} |X_s^{\varepsilon,\delta} - x|^2 \Big] \\
&\le Kh.
\end{aligned} \tag{4.3.23}$$

Combining (4.3.19), (4.3.21) and (4.3.23), due to *Cauchy-Schwarz inequality*, we obtain for some $K > 0$

$$\begin{aligned}
\xi_h &= \frac{1}{h} \bar{\mathbb{E}}\Big[\int_t^{t+h} U^{\varepsilon,\delta}(r, X_r^{\varepsilon,\delta}) dr \mathbf{1}_{\{\zeta > h\}} \Big] + \bar{\mathbb{E}}\Big[\int_t^{t+h} U^{\varepsilon,\delta}(r, X_r^{\varepsilon,\delta}) dr \mathbf{1}_{\{\zeta \le h\}} \Big] \\
&\le -\tau \bar{\mathbb{P}}(\zeta > h) + \frac{1}{h} \sqrt{\bar{\mathbb{P}}(\zeta \le h)} \sqrt{h} \sqrt{\bar{\mathbb{E}}\Big[\int_t^{t+h} |U^{\varepsilon,\delta}(r, X_r^{\varepsilon,\delta})|^2 dr \Big]} \\
&\le -\tau(1 - Kh) + K\sqrt{h} \sqrt{\bar{\mathbb{E}}\Big[1 + \sup_{t \le s \le t+h} |X_s^{\varepsilon,\delta}|^4 \Big]},
\end{aligned} \tag{4.3.24}$$

which implies that if (4.3.20) holds, there exists $\tau_0 > 0$ and $h_1 > 0$ such that for all $h < h_1$, we have $\xi_h \leq -\tau_0$.
Due to (4.3.14), we conclude that

$$\begin{aligned}
\hat{Y}_i^{\varepsilon,\delta}(t) &= \bar{Y}_i^{\varepsilon,\delta}(t) - \varphi(t, X_t^{\varepsilon,\delta}) \\
&= \bar{Y}_i^{\varepsilon,\delta}(t) - u_i^{\varepsilon,\delta}(t,x) \\
&= \bar{Y}_i^{\varepsilon,\delta}(t) - Y_i^{\varepsilon,\delta}(t) \geq 0,
\end{aligned}$$

and consequently

$$\begin{aligned}
&0 \leq \frac{1}{h}\hat{Y}_i^{\varepsilon,\delta}(t) \\
&= \frac{1}{h}\bar{\mathbb{E}}\Big[\int_t^{t+h} \Theta^{\varepsilon,\delta}(r, X_r^{\varepsilon,\delta}) \\
&+ f_i\Big(r, X_r^{\varepsilon,\delta}, (\varphi(r, X_r^{\varepsilon,\delta}) + \hat{Y}_i^{\varepsilon,\delta}(r), \tilde{u}_i^{\varepsilon,\delta}(r, X_r^{\varepsilon,\delta})), \\
&\sqrt{\varepsilon}\nabla_x\varphi(r, X_r^{\varepsilon,\delta})\sigma(r, X_r^{\varepsilon,\delta}, u^{\varepsilon,\delta}(r, X_r^{\varepsilon,\delta})) + \hat{Z}_i^{\varepsilon,\delta}(r), \\
&\int_{\mathbb{R}^d}(\Gamma^\delta(r, X_r^{\varepsilon,\delta}, z) + U^{\varepsilon,\delta}(r,z))\frac{\gamma^\delta(z)}{\delta}\nu(dz)\Big)dr\Big].
\end{aligned} \tag{4.3.25}$$

Therefore, for $h > 0$ small enough, (4.3.24), (4.3.25), the sublinear growth of f, (4.3.17), (4.3.18) and (4.3.7) yield that there exists $K > 0$, that may change from line to line, such that

$$\begin{aligned}
\tau_0 &\leq \Big|\frac{1}{h}\hat{Y}_i^{\varepsilon,\delta}(t) - \xi_h\Big| \\
&\leq K\bar{\mathbb{E}}\Big[\frac{1}{h}\int_t^{t+h}\Big(|\hat{Y}_i^{\varepsilon,\delta}(r)| + |\hat{Z}_i^{\varepsilon,\delta}(r)| + \Big(\frac{1}{h}\int_{\mathbb{R}^d}|\hat{V}_i^{\varepsilon,\delta}(r,z)\frac{\nu(dz)}{\delta}dr\Big)\Big] \\
&\leq K\Big(\sup_{t\leq r\leq t+h}\bar{\mathbb{E}}[|\hat{Y}_i^{\varepsilon,\delta}(r)|] + \Big(\frac{1}{h}\bar{\mathbb{E}}\Big[\int_t^{t+h}|\hat{Z}_i^{\varepsilon,\delta}(r)|^2 dr\Big]\Big)^{1/2} \\
&+ \Big(\frac{1}{h}\bar{\mathbb{E}}\Big[\int_t^{t+h}\int_{\mathbb{R}^d}|\hat{V}_i^{\varepsilon,\delta}(r)|^2\frac{\nu(dz)}{\delta}dr\Big]\Big)^{1/2}\Big) \\
&\leq K(\sqrt{h} + h^{1/4} + h^{1/4}) \\
&\leq Kh^{1/4},
\end{aligned}$$

which is a contradiction for $h > 0$ small enough. In conclusion (4.3.19) cannot hold and we conclude that $u^{\varepsilon,\delta}$ is a viscosity subsolution of (4.3.1).

□

Remark 4.3.4. *For every $\varepsilon, \delta > 0$ the function $u^{\varepsilon,\delta}(t,x) := Y_t^{\varepsilon,\delta}$, $(t,x) \in [T', T] \times \mathbb{R}^d$ in the context of the previous theorem is the unique viscosity solution of the terminal value* *problem (4.3.1) in the class of the functions that satisfy the following growth condition*

$$\lim_{|x|\to\infty} |u^{\varepsilon,\delta}(t,x)|e^{-c\ln^2|x|} = 0, \quad T' \le t \le T, c \ge 0.$$

*The proof follows exactly as in Barles et al. (1996)-**Theorem 3.5**.*

Remark 4.3.5. *For every $\varepsilon, \delta > 0$, $x \in \mathbb{R}^d$ and $t \in [T', T]$ with $T' < T$ given by **Theorem 4.2.1**, the method of decoupling fields ask not only for a solution of (4.2.8) but also for a measurable function $u^{\varepsilon,\delta}$ such that the backward process $Y_s^{t,x,\varepsilon,\delta} = u^{\varepsilon,\delta}(s, X_s^{t,x,\varepsilon,\delta})$ for every $s \in [t,T]$. The function $u^{\varepsilon,\delta}$ that is a viscosity solution of the terminal value problem (4.3.1) is a candidate for a decoupling field of (4.2.8). In the case $u^{\varepsilon,\delta}$ is a classic solution of (4.3.1) it is a well-known fact that $T' = 0$ (**Theorem 4.3.1**). Nevertheless, the question if there exists a global solution of (4.2.8), i.e. $T' = 0$, in the conditions of **Theorem 4.3.2** is not answered yet. The author believes that it is an interesting problem to use the function $u^{\varepsilon,\delta}$ viscosity solution of (4.3.1) as a candidate for a decoupling field and verify if it is possible in the conditions of **Theorem 4.3.2** to assure the existence and uniqueness of a global solution for the FBSDE system (4.2.8).*

The following result concerns the uniform convergence of viscosity solutions for the terminal value problem (4.3.1).

Proposition 4.3.4. *For every $0 < \varepsilon, \delta \le 1$, under **Condition 4.2.2**, let $u^{\varepsilon,\delta}$ be a viscosity solution of (4.3.1) converging uniformly on the compact sets of $[T', T] \times \mathbb{R}^d$ to u. Then u is a viscosity solution of the following terminal value problem of first order*

$$\begin{cases} \partial_t u + \langle b(t,x,u(t,x)), \nabla_x u(t,x)\rangle + f(t,x,u(t,x),0,0) = 0, \\ u(T,x) = g(x), \quad x \in \mathbb{R}^d, t \in [T', T]. \end{cases} \tag{4.3.26}$$

Proof. The proof is immediate, since the definition of viscosity solution for the PIDE (4.3.1) was given in order to be preserved by uniform limit operations. For every $\varepsilon, \delta > 0$, let $u^{\varepsilon,\delta}$ be a viscosity solution of (4.3.1). We prove the viscosity subsolution property for u. The viscosity supersolution property of u follows with analogous argument.

Since $u^{\varepsilon,\delta} \to u$ as $\varepsilon, \delta \to 0$, uniformly on compact sets of $[T', T] \times \mathbb{R}^d$ and $u^{\varepsilon,\delta} \in C([T', T] \times \mathbb{R}^d, \mathbb{R}^n)$, we have that u is a continuous function in $[T', T] \times \mathbb{R}^d$. Since $u_i^{\varepsilon,\delta}(T, x) \le g(x)$ for all $i \in \{1, ..., n\}$ and for all $x \in \mathbb{R}^d$, sending $\varepsilon, \delta \to 0$ we concude that $u(T, x) \le g(x)$ for all $x \in \mathbb{R}^d$.

For any $i \in \{1, ..., n\}$ and $\varphi \in C^{1,2}([T', T] \times \mathbb{R}^d)$, whenever $(t, x) \in [T', T] \times \mathbb{R}^d$ is a strict local point of maximum of $u_i - \varphi$, we show that

$$-\partial_t \varphi(t,x) - f_i(t,x,u(t,x),0,0) \le 0. \tag{4.3.27}$$

1. We show that there exists a sequence $(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n})_{n\in\mathbb{N}} \subset (t^{\varepsilon,\delta}, x^{\varepsilon,\delta})_{\varepsilon,\delta>0}$ such that, for every $n \in \mathbb{N}$, $(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) \in [T', T] \times \mathbb{R}^d$; $(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n})$ is a point of strict local maximum of $u^{\varepsilon_n,\delta_n} - \varphi$; (t, x) is the limit of $((t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}))_{n\in\mathbb{N}}$ and $u(t, x) - \varphi(t, x)$ is the limit of $(u^{\varepsilon_n,\delta_n}(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) - \varphi(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}))_{n\in\mathbb{N}}$.
   Since (t, x) is a strict local maximum of $u_i - \varphi$, let $h > 0$ such that $T' < t-h$ and for every $(s, x) \in [t - h, t + h] \times \mathrm{cl}(B_h(x))$ we have $u_i(s, x) < \varphi(s, x)$. For every $\varepsilon, \delta > 0$, since $u^{\varepsilon,\delta} - \varphi$ is continuous, let $(t^{\varepsilon,\delta}, x^{\varepsilon,\delta}) \in [t - h, t + h] \times B_h(x)$ such that

   $$u^{\varepsilon,\delta}(t^{\varepsilon,\delta}, x^{\varepsilon,\delta}) = \sup_{[t-h,t+h]\times B_h(x)} (u^{\varepsilon,\delta} - \varphi).$$

   Due to the compactness of $[t - h, t + h] \times \mathrm{cl}(B_h(x))$, there exists a limit point $(\bar{t}, \bar{x}) \in [t-h, t+h]\times \mathrm{cl}(B_h(x))$ of $(t^{\varepsilon,\delta}, x^{\varepsilon,\delta})_{\varepsilon,\delta>0}$ as $\varepsilon, \delta \to 0$. Moreover, let $(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n})_{n\in\mathbb{N}} \subset (t^{\varepsilon,\delta}, x^{\varepsilon,\delta})_{\varepsilon,\delta>0}$ such that $(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) \to (\bar{t}, \bar{x})$ as $n \to \infty$.
   We have the following estimate, for every $n \in \mathbb{N}$

   $$u^{\varepsilon_n,\delta_n}(t, x) - \varphi(t, x) \le u^{\varepsilon_n,\delta_n}(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) - \varphi(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}). \tag{4.3.28}$$

   Given $\mu > 0$, due to (4.3.11) and the fact that φ is continuous there exists $\eta > 0$ such that $|t^{\varepsilon_n,\delta_n} - \bar{t}| < \eta$ and $|x^{\varepsilon_n,\delta_n} - \bar{x}| < \eta$ implies, for all $n \in \mathbb{N}$ large enough,

   $$\begin{aligned} |u^{\varepsilon_n,\delta_n}(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) - u^{\varepsilon_n,\delta_n}(\bar{t}, \bar{x})| &< \frac{\mu}{2}, \\ |\varphi(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) - \varphi(\bar{t}, \bar{x})| &< \frac{\mu}{2}. \end{aligned}$$

   Since $(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) \to (\bar{t}, \bar{x})$, as $n \to \infty$, there exist $p_1 \in \mathbb{N}$ such that for $n \ge p_1$ we have

   $$|t^{\varepsilon_n,\delta_n} - \bar{t}| < \eta \quad \text{and} \quad |x^{\varepsilon_n,\delta_n} - \bar{x}| < \eta.$$

   Due to uniform convergence of $u^{\varepsilon,\delta}$ to u in $[t - h, t + h] \times \mathrm{cl}(B_h(x))$, it follows that there exists $p_2 \in \mathbb{N}$ such that, for all $n \ge p_2$ and $(s, y) \in [t - h, t + h] \times \mathrm{cl}(B_h(x))$,

   $$|u^{\varepsilon_n,\delta_n}(s, x) - u(s, x)| < \frac{\mu}{2}.$$

   Therefore, for $n \ge p_1 \vee p_2$ we have

   $$\begin{aligned} |u^{\varepsilon_n,\delta_n}(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) - u(\bar{t}, \bar{x})| &\le |u^{\varepsilon_n,\delta_n}(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) - u^{\varepsilon_n,\delta_n}(\bar{t}, \bar{x})| + |u^{\varepsilon_n,\delta_n}(\bar{t}, \bar{x}) - u(\bar{t}, \bar{x})| \\ &\le \mu. \end{aligned}$$

   Hence, (4.3.28) yields, for every $\mu > 0$ and $n \ge n_0$ with $n_0 \in \mathbb{N}$ sufficiently large,

   $$\begin{aligned} u(t, x) - \varphi(t, x) - \frac{\mu}{2} &\le u^{\varepsilon_n,\delta_n}(t, x) - \varphi(t, x) \\ &\le u^{\varepsilon_n,\delta_n}(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) - \varphi(t^{\varepsilon_n,\delta_n}, x^{\varepsilon_n,\delta_n}) \\ &\le u(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x}) + \mu. \end{aligned}$$

   Since (t, x) is a strict local maximum, it follows that $(t, x) = (\bar{t}, \bar{x})$.

2. By definition of viscosity subsolution for (4.3.1), we have

$$\begin{aligned}
&-\partial_t\varphi(t,x) - \mathcal{L}^{\varepsilon,\delta}\varphi(t,x)\\
&- f_i(t,x,u^{\varepsilon,\delta}(t,x),\sqrt{\varepsilon}\nabla_x\varphi(t,x)\sigma(t,x,u^{\varepsilon,\delta}(t,x)),\mathcal{J}^{\delta}\varphi(t,x)) \le 0.
\end{aligned} \tag{4.3.29}$$

We observe that $\varphi \in C^{1,2}([T',T]\times\mathbb{R}^d)$ and therefore, for the sequence $(t^{\varepsilon_n,\delta_n},x^{\varepsilon_n,\delta_n})_{n\in\mathbb{N}}$ of the previous point we have the following convergences, as $n\to\infty$,

$$\begin{aligned}
\partial_t\varphi(t^{\varepsilon_n,\delta_n},x^{\varepsilon_n,\delta_n}) &\to \partial_t\varphi(t,x),\\
\nabla_x\varphi(t^{\varepsilon_n,\delta_n},x^{\varepsilon_n,\delta_n}) &\to \nabla_x\varphi(t,x), \text{ and}\\
\nabla_x^2\varphi(t^{\varepsilon_n,\delta_n},x^{\varepsilon_n,\delta_n}) &\to \nabla_x^2\varphi(t,x.)
\end{aligned}$$

We show that for $(t,x)\in[T',T]\times\mathbb{R}^d$

$$\lim_{\delta\to 0}\mathcal{J}^{\delta}\varphi(t,x)=0, \qquad \text{for all } \varphi\in C^{1,2}([T',T]\times\mathbb{R}^d),$$

This follows from the following estimate.
Fatou's lemma, *the Cauchy-Schwarz inequality* and **Condition 4.2.2** yield

$$\begin{aligned}
&\limsup_{\delta\to 0}|\mathcal{J}^{\delta}\varphi(t,x)|\\
&\le \int_{\mathbb{R}^d}\limsup_{\delta\to 0}\Big|\frac{\varphi(t,x+\delta\beta(x,z))-\varphi(t,x)}{\delta}\Big||\gamma^{\delta}(z)|\nu(dz)\\
&\le \int_{\mathbb{R}^d}|\langle\nabla_x\varphi(t,x),\beta(x,z)\rangle|\limsup_{\delta\to 0}|\gamma^{\delta}(z)|\nu(dz)\\
&\le \sup_{(s,y)\in[t,t+h]\times\mathrm{cl}(B_h(x))}|\nabla\varphi(s,y)|\limsup_{\delta\to 0}\kappa(\delta)\Big(\int_{\mathbb{R}^d}(1\wedge|z|^2)\nu(dz)\Big)^{1/2}||\beta(x,.)||_{L^2(\mathbb{R}^d,\nu)}\\
&= 0.
\end{aligned}$$

Due to the continuity of the coefficients of $\mathcal{L}^{\varepsilon,\delta}$, the continuity of f and the convergences pointed out before, as $\varepsilon,\delta\to 0$,

$$\mathcal{L}^{\varepsilon_n,\delta_n}\varphi(t^{\varepsilon_n,\delta_n},x^{\varepsilon_n,\delta_n}) \to \langle b(t,x,\varphi(t,x)),\nabla_x\varphi(t,x)\rangle,$$

and therefore, passing to the limit in (4.3.29), we obtain (4.3.27).

□

4.4 The almost sure convergence

We state our result concerning the almost-sure convergence of (4.3.4) as $\varepsilon, \delta \to 0$. If (4.3.1) takes the form of the terminal value problem for the backward fractal Burgers equations (4.3.6) the following result characterizes probabilistically the vanishing limit of the local viscosity, $\varepsilon \to 0$, and the nonlocal viscosity, $\delta \to 0$, for the velocity of the fluid modelled by these equations.

Theorem 4.4.1 (**Almost sure convergence**).

1. *For every $0 < \varepsilon \leq 1$, $t \in [T', T]$ and $x \in \mathbb{R}^d$, under* ***Condition 4.2.2****, the solution $(X_s^{t,x,\varepsilon}, Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}, V_s^{t,x,\varepsilon})_{t\leq s\leq T}$, given by* ***Theorem 4.2.1****, of the FBSDE system*

$$\begin{cases} X_s^{t,x\varepsilon} & = x + \displaystyle\int_t^s b(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon})dr \\ & \quad + \sqrt{\varepsilon} \displaystyle\int_t^s \sigma(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon})dB_r + \varepsilon \int_t^s \int_{\mathbb{R}^d} \beta(X_{r-}^{t,x,\varepsilon}, z)\tilde{N}^{\frac{1}{\varepsilon}}(dr, dz), \\ Y_s^{t,x,\varepsilon} & = g(X_T^{t,x,\varepsilon}) + \displaystyle\int_s^T f\Big(r, X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon}, Z_r^{t,x,\varepsilon}, \int_{\mathbb{R}^d} V_{r-}^{t,x,\varepsilon}(z)\gamma^{\varepsilon}(z)\frac{\nu(dz)}{\varepsilon}\Big)dr \\ & \quad - \displaystyle\int_s^T Z_r^{t,x,\varepsilon}dB_r - \int_s^T \int_{\mathbb{R}^d} V_{r-}^{t,x,\varepsilon}(z)\tilde{N}^{\frac{1}{\varepsilon}}(dr, dz), \quad t \leq s \leq T, \end{cases} \tag{4.4.1}$$

converges in $\mathcal{S}^2(t, T, \mathbb{R}^d) \times \mathcal{S}^2(t, T, \mathbb{R}^n) \times \mathcal{H}^2(t, T, \mathbb{R}^{n\times d}) \times \mathcal{H}_\nu^2(t, T, \mathbb{R}^n)_{t\leq s\leq T}$ to $(X_s^0, Y_s^0, 0, 0)_{t\leq s\leq T'}$ where $(X_s^0, Y_s^0)_{t\leq s\leq T}$ solves the following two-point boundary value problem of ordinary differential equations:

$$\begin{cases} \dot{X}_s & = b(s, X_s, Y_s), \\ \dot{Y}_s & = -f(s, X_s, Y_s, 0), \quad t \leq s \leq T, \\ X_t & = x, \\ Y_T & = g(X_T). \end{cases} \tag{4.4.2}$$

2. *Let us fix $0 < \delta \leq 1$. Then as $\varepsilon \to 0$, $(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, Z_s^{\varepsilon,\delta}, V_s^{\varepsilon,\delta})_{t\leq s\leq T}$ converges in $\mathcal{S}^2(t, T, \mathbb{R}^d) \times \mathcal{S}^2(t, T, \mathbb{R}^n) \times \mathcal{H}^2(t, T, \mathbb{R}^{n\times d}) \times \mathcal{H}_\nu^2(t, T, \mathbb{R}^n)_{t\leq s\leq T}$ to $(X_s^{0,\delta}, Y_s^{0,\delta}, 0, V_s^{0,\delta})_{t\leq s\leq T'}$ solution of (4.3.4) with $\varepsilon = 0$. Moreover, the function $u^{\varepsilon,\delta}(t, x) := Y_t^{t,x,\varepsilon,\delta}$, viscosity solution of (4.3.1), converges uniformly in compact sets of $[T', T] \times \mathbb{R}^d$ to $u^{0,\delta}$, viscosity solution of (4.3.1) with $\varepsilon = 0$.*

3. *An analogous result holds if $\delta \to 0$.*

4. *As $\varepsilon, \delta \to 0$, the limit function $u(t, x) = Y_t^{t,x}$ of $u^{\varepsilon,\delta}$ is a viscosity solution of the first order terminal value problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \langle b(t, x, u(t, x)), \nabla_x u(t, x)\rangle + f(t, x, u(t, x), 0, 0) = 0 \\ u(T, x) = g(x), \quad x \in \mathbb{R}^d, t \in [T', T]. \end{cases} \tag{4.4.3}$$

Furthermore, if $u \in C_b^{1,1}([0,T]\times\mathbb{R}^d)$, continuosuly differentiable with bounded derivatives, the function u is the unique classical solution of (4.4.3).

Proof.

1. By **Theorem 4.2.1** let $T' < T$ and some arbitrarily fixed $t \in [T', T]$ such that $(X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon, V_r^\varepsilon)_{t\le r\le T} \in \mathcal{M}^2[t,T]$ is the unique solution of (4.4.1). For sake of simplicity, we drop the dependence on the initial condition $x \in \mathbb{R}^d$ and initial time $t \in [T', T]$ for the solution process of (4.4.1). The deterministic function $(X_s^0, Y_s^0)_{t\le s\le T}$ is the unique continuous solution of (4.4.2).
   We use the notation, for all $\varepsilon > 0$, $r \in [t,T]$ and $z \in \mathbb{R}^d$,

$$\begin{aligned}
b^\varepsilon(r) &= b(r, X_r^\varepsilon, Y_r^\varepsilon),\\
b^0(r) &= b(r, X_r^0, Y_r^0),\\
\sigma^\varepsilon(r) &= \sigma(r, X_r^\varepsilon, Y_r^\varepsilon),\\
\beta^\varepsilon(r,z) &= \beta(X_r^\varepsilon, z),\\
f^\varepsilon(r) &= f\Big(r, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon, \int_{\mathbb{R}^d} V_r^\varepsilon(z)\gamma^\varepsilon(z)\frac{1}{\varepsilon}\nu(dz)\Big),\\
f^0(r) &= f(r, X_r^0, Y_r^0, 0, 0).
\end{aligned}$$

   Applying *Itô's formula* to $|X_s^\varepsilon - X_s^0|^2$ for $s \in [t,T]$ and taking expectations we derive

$$\begin{aligned}
\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|X_s^\varepsilon - X_s^0|^2\Big] &\le \bar{\mathbb{E}}\Big[\int_t^T \varepsilon|\sigma^\varepsilon(r)|^2 dr\Big] + 2\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}\int_t^s |\langle X_r^\varepsilon - X_r^0, b^\varepsilon(r) - b^0(r)\rangle| dr\Big]\\
&+ 2\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}\Big|\int_t^s \langle X_r^\varepsilon - X_r^0, \sqrt{\varepsilon}\sigma^\varepsilon(r)dB_r\rangle\Big|\Big]\\
&+ \bar{\mathbb{E}}\Big[\sup_{t\le s\le T}\Big|\int_t^s\int_{\mathbb{R}^d}(\varepsilon|\beta^\varepsilon(r,z)|^2 + 2\langle X_{r-}^\varepsilon - X_r^0, \varepsilon\beta^\varepsilon(r,z)\rangle)\tilde{N}^{\frac{1}{\varepsilon}}(drdz)\Big|\Big]\\
&+ \bar{\mathbb{E}}\Big[\sup_{t\le s\le T}\int_t^s\int_{\mathbb{R}^d}\varepsilon^2|\beta^\varepsilon(r)|^2\frac{1}{\varepsilon}\nu(dz)dr\Big]. \qquad (4.4.4)
\end{aligned}$$

   Straightforward estimates and *Burkholder-Davis-Gundy's inequalities* (**Proposition B.3.3**) imply that there exists a constant $C_1 > 0$, that may change from line to line, such that

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\Big|\int_s^T \langle X_r^\varepsilon - X_r^0, \sqrt{\varepsilon}\sigma^\varepsilon(r)dB_r\rangle\Big|\Big] \\
&\leq C_1\bar{\mathbb{E}}\Big[\Big(\int_t^T |\langle X_r^\varepsilon - X_r^0, \sqrt{\varepsilon}\sigma^\varepsilon(r)\rangle|^2 dr\Big)^{1/2}\Big] \\
&\leq C_1\bar{\mathbb{E}}\Big[\Big(\int_t^T \varepsilon|\sigma^\varepsilon(r)|^2|X_r^\varepsilon - X_r^0|^2 dr\Big)^{1/2}\Big] \\
&\leq C_1\sqrt{\varepsilon}\bar{\mathbb{E}}\Big[\Big(\sup_{t\leq s\leq T}|X_s^\varepsilon - X_s^0|^2\int_t^T |\sigma^\varepsilon(r)|^2 dr\Big)^{1/2}\Big] \\
&\leq C_1\frac{\sqrt{\varepsilon}}{2}\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}|X_s^\varepsilon - X_s^0|^2\Big] + C_1\frac{\sqrt{\varepsilon}}{2}\bar{\mathbb{E}}\Big[\int_t^T |\sigma^\varepsilon(r)|^2 dr\Big]. \qquad (4.4.5)
\end{aligned}$$

Due to *Burkholder-Davis-Gundy's inequalities*, *Itô's isometry* and *the Cauchy-Schwarz inequality* there exists $C_2 > 0$ such that for every $\alpha_1 > 0$ we have

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\Big|\int_t^T\int_{\mathbb{R}^d}\langle X_{r-}^\varepsilon - X_r^0, \varepsilon\beta^\varepsilon(r,z)\rangle\tilde{N}^{\frac{1}{\varepsilon}}(dr,dz)\Big|\Big] \\
&= C_2\bar{\mathbb{E}}\Big[\Big(\int_t^T\int_{\mathbb{R}^d}\varepsilon^2|\langle\beta^\varepsilon(r,z), X_{r-}^\varepsilon - X_r^0\rangle|^2 N^{\frac{1}{\varepsilon}}(dr,dz)\Big)^{1/2}\Big] \\
&\leq C_2\varepsilon\bar{\mathbb{E}}\Big[\Big(\int_t^T\int_{\mathbb{R}^d}|\beta^\varepsilon(r,z)|^2|X_{r-}^\varepsilon - X_r^0|^2 N^{\frac{1}{\varepsilon}}(drdz)\Big)^{1/2}\Big] \\
&\leq C_2\varepsilon\bar{\mathbb{E}}\Big[\sqrt{\sup_{t\leq s\leq T}|X_s^\varepsilon - X_s^0|^2}\sqrt{\int_t^T\int_{\mathbb{R}^d}|\beta^\varepsilon(r,z)|^2 N^{\frac{1}{\varepsilon}}(dr,dz)}\Big] \\
&\leq C_2\varepsilon\alpha_1\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}|X_s^\varepsilon - X_s^0|^2\Big] + \frac{C_2}{\alpha_1}\bar{\mathbb{E}}\Big[\int_t^T |\beta^\varepsilon(r,z)|^2\nu(dz)dr\Big]. \qquad (4.4.6)
\end{aligned}$$

Combining (4.4.4), (4.4.5) and (4.4.6) implies that, for $\alpha_1 = \frac{1}{2\varepsilon C_2}$,

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}|X_s^\varepsilon - X_s^0|^2\Big] \\
&\leq \varepsilon||\sigma||^2_{L^\infty([t,T])}(T-t) + \bar{\mathbb{E}}\Big[\int_t^T \sup_{t\leq r\leq s}|X_r^\varepsilon - X_r^0|^2 ds\Big] \\
&+ \bar{\mathbb{E}}\Big[\int_t^T \sup_{t\leq r\leq s}|b^\varepsilon(r) - b^0(r)|^2 ds\Big] + \frac{C_1\sqrt{\varepsilon}}{2}\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}|X_s^\varepsilon - X_s^0|^2\Big] \\
&+ \frac{C_1\sqrt{\varepsilon}}{2}||\sigma||^2_{L^\infty([t,T])} + 2\varepsilon||\beta||^2_{L^2([t,T]\times\mathbb{R}^d, ds\otimes\nu)} \\
&+ \frac{1}{2}\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}|X_s^\varepsilon - X_s^0|^2\Big] + 2\varepsilon C_2^2||\beta||^2_{L^2([t,T]\times\mathbb{R}^d, ds\otimes\nu)}.
\end{aligned}$$

In conclusion, fixed $\varepsilon_0 < \frac{1}{C_1^2}$, the Lipschitz continuity of the function b yields that there exists a constant $C_3 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ we have

$$\begin{aligned}&\Big(\frac{1}{2} - \frac{C_1\sqrt{\varepsilon}}{2}\Big)\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|X_s^\varepsilon - X_s^0|^2\Big]\\ &\le C_3\Big(1 + \bar{\mathbb{E}}\Big[\int_t^T \sup_{t\le s\le r}|X_s^\varepsilon - X_s^0|^2 dr\Big] + \bar{\mathbb{E}}\Big[\int_t^T \sup_{t\le s\le r}|Y_s^\varepsilon - Y_s^0|^2 dr\Big]\Big). \qquad (4.4.7)\end{aligned}$$

Given $\rho > 0$, applying *Itô's formula* to $e^{\rho s}|Y_s^\varepsilon - Y_s^0|^2$, $s \in [t,T]$ we obtain

$$\begin{aligned}&e^{\rho s}|Y_s^\varepsilon - Y_s^0|^2 + \rho\int_s^T e^{\rho u}|Y_u^\varepsilon - Y_u^0|^2 du\\ &+ \int_s^T e^{\rho u}|Z_u^\varepsilon|^2 du + \int_s^T\int_{\mathbb{R}^d} e^{\rho u}|V_u^\varepsilon(z)|^2\frac{1}{\varepsilon}\nu(dz)du\\ &= e^{\rho T}|g(X_T^\varepsilon) - g(X_T^0)|^2 - 2\int_s^T e^{\rho u}\langle Y_u^\varepsilon - Y_u^0, -f^\varepsilon(u) + f^0(u)\rangle du\\ &- 2\int_s^T e^{\rho u}\langle Y_u^\varepsilon - Y_u^0, Z_s^\varepsilon dB_u\rangle\\ &- 2\int_s^T\int_{\mathbb{R}^d} e^{\rho u}\langle Y_{u-}^\varepsilon - Y_u^0, V_u^\varepsilon(z)\rangle N^{\frac{1}{\varepsilon}}(dz,du), \quad t\le s\le T. \qquad (4.4.8)\end{aligned}$$

The following estimate is straightforward,

$$\begin{aligned}&\sup_{t\le s\le T} e^{\rho s}|Y_s^\varepsilon - Y_s^0|^2 + \rho\int_s^T e^{\rho u}|Y_u^\varepsilon - Y_u^0|^2 du\\ &+ \int_s^T e^{\rho u}|Z_u^\varepsilon|^2 du + \int_s^T\int_{\mathbb{R}^d} e^{\rho u}|V_u^\varepsilon(z)|^2\frac{1}{\varepsilon}\nu(dz)du\\ &\le e^{\rho T}|g(X_T^\varepsilon) - g(X_T^0)|^2 + 2\int_t^T e^{\rho u}|\langle Y_u^\varepsilon - Y_u^0, -f^\varepsilon(u) + f^0(u)\rangle du|\\ &+ 2\sup_{t\le s\le T}\Big|\int_s^T e^{\rho u}\langle Y_u^\varepsilon - Y_u^0, Z_s^\varepsilon dB_u\rangle\Big|\\ &+ 2\sup_{t\le s\le T}\Big|\int_s^T\int_{\mathbb{R}^d} e^{\rho u}\langle Y_u^\varepsilon - Y_u^0, V_u^\varepsilon(z)\rangle\tilde{N}^{\frac{1}{\varepsilon}}(dz,du)\Big|\\ &+ 2\int_t^T\int_{\mathbb{R}^d} e^{\rho u}|\langle Y_u^\varepsilon - Y_u^0, V_u^\varepsilon(z)\rangle|\frac{1}{\varepsilon}\nu(dz)du.\end{aligned}$$

By *Burkholder-Davis-Gundy's inequalities* (**Proposition B.3.3**) we have that there exists $C_4 > 0$, that may change from line to line, such that, for all $\alpha_2 > 0$, we have

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\le s\le T} e^{\rho s}|Y^\varepsilon_s - Y^0_s|^2\Big] + \rho\bar{\mathbb{E}}\Big[\int_s^T e^{\rho u}|Y^\varepsilon_u - Y^0_u|^2 du\Big] \\
&+ \bar{\mathbb{E}}\Big[\int_s^T e^{\rho u}|Z^\varepsilon_u|^2 du\Big] + \bar{\mathbb{E}}\Big[\int_s^T\int_{\mathbb{R}^d} e^{\rho u}|V^\varepsilon_u(z)|^2\frac{1}{\varepsilon}\nu(dz)du\Big] \\
&\le \bar{\mathbb{E}}\Big[e^{\rho T}|g(X^\varepsilon_T) - g(X^0_T)|^2\Big] + 2\bar{\mathbb{E}}\Big[\int_t^T e^{\rho u}|Y^\varepsilon_u - Y^0_u||f^\varepsilon(u) - f^0(u)|du\Big] \\
&+ C_4\bar{\mathbb{E}}\Big[\Big(\int_t^T e^{2\rho u}|Y^\varepsilon_u - Y^0_u|^2|Z^\varepsilon_u|^2 du\Big)^{1/2}\Big] \\
&+ C_4\bar{\mathbb{E}}\Big[\Big(\int_t^T\int_{\mathbb{R}^d} e^{2\rho u}|Y^\varepsilon_u - Y^0_u|^2|V^\varepsilon_u(z)|^2 N^{\frac{1}{\varepsilon}}(dz,du)\Big)^{1/2}\Big] \\
&+ C_4\bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d}|V^\varepsilon_s(z)|^2\frac{1}{\varepsilon}\nu(dz)ds\Big] \\
&\le \bar{\mathbb{E}}\Big[e^{\rho T}|g(X^\varepsilon_T) - g(X^0_T)|^2\Big] + 2\bar{\mathbb{E}}\Big[\int_t^T e^{\rho u}|Y^\varepsilon_u - Y^0_u||f^\varepsilon(u) - f^0(u)|du\Big] \\
&+ C_4\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|Y^\varepsilon_s - Y^s_0|\Big(\int_s^T e^{2\rho u}|Z^\varepsilon_u|^2 du\Big)^{\frac{1}{2}}\Big] \\
&+ C_4\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|Y^\varepsilon_s - Y^s_0|\Big(\int_t^t\int_{\mathbb{R}^d} e^{2\rho u}|V^\varepsilon_u(z)|^2 N^{\frac{1}{\varepsilon}}(du,dz)\Big)^{1/2}\Big] \\
&\le \bar{\mathbb{E}}\Big[e^{\rho T}|g(X^\varepsilon_T) - g(X^0_T)|^2\Big] + 2\bar{\mathbb{E}}\Big[\int_t^T e^{\rho u}|Y^\varepsilon_u - Y^0_u||f^\varepsilon(u) - f^0(u)|du\Big] \\
&+ \frac{C_4}{\alpha_2}\bar{\mathbb{E}}\Big[\sup_{t\le s\le T} e^{\rho s}|Y^\varepsilon_s - Y^0_s|^2\Big] + C_4\alpha_2\bar{\mathbb{E}}\Big[\int_t^T e^{2\rho u}|Z^\varepsilon_u|^2 du\Big] \\
&+ C_4\alpha_2\bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d} e^{2\rho u}|V^\varepsilon_u(z)|^2\frac{1}{\varepsilon}\nu(dz)du\Big].
\end{aligned}$$

We choose $\alpha_2 = 2C_4$ and therefore, for some $C_5 = C_5(\rho, T)$,

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\le s\le T} e^{\rho s}|Y^\varepsilon_s - Y^0_s|^2\Big] \\
&\le C_5\Big(\bar{\mathbb{E}}\Big[e^{\rho T}|g(X^\varepsilon_T) - g(X^\varepsilon_0)|^2\Big] + ||Z^\varepsilon||^2_{\rho,\mathcal{H}^2(t,T,\mathbb{R}^{n\times d})} + \Big|\Big|\frac{V^\varepsilon}{\varepsilon}\Big|\Big|^2_{\mathcal{H}^2_{\rho,\nu}(t,T,\mathbb{R}^n)} \\
&+ 2\bar{\mathbb{E}}\Big[\int_t^T e^{\rho u}|Y^\varepsilon_u - Y^0_u||f^\varepsilon(u) - f^0(u)|du\Big]\Big)
\end{aligned}$$

The previous estimate, the Lipschitz continuity of f and (4.4.7) imply that there exists some constant $C_6 = C_6(K_1, K_3, T, T') > 0$, where $K_1, K_3 > 0$ are given in **Condition**

4.2.2, such that

$$
\begin{aligned}
&||Y^\varepsilon - Y^0||^2_{\rho,\mathcal{S}^2(t,T,\mathbb{R}^n)} + \bar{\mathbb{E}}\Big[\int_s^T e^{\rho u}|Z_u^\varepsilon|^2 du\Big] + \bar{\mathbb{E}}\Big[\int_s^T \int_{\mathbb{R}^d} e^{\rho u}|V_u^\varepsilon(z)|^2 \frac{1}{\varepsilon}\nu(dz)du\Big] \\
&\le C_6\Big(\bar{\mathbb{E}}\Big[e^{\rho T}|g(X_T^\varepsilon) - g(X_T^\varepsilon)|^2\Big] + \rho\bar{\mathbb{E}}\Big[\int_t^T e^{\rho u}|Y_u^\varepsilon - Y_u^0|^2 du\Big] \\
&+ \frac{1}{\rho}\bar{\mathbb{E}}\Big[\int_t^T e^{\rho u}|Z_u^\varepsilon|^2 du\Big] + \frac{1}{\rho}\bar{\mathbb{E}}\Big[\int_t^T \int_{\mathcal{X}} |V_u^\varepsilon(z)|^2 \frac{1}{\varepsilon}\nu(dz)\Big]\Big) \qquad (4.4.9)
\end{aligned}
$$

Observing that g is bounded we conclude, for some constants $C_7 > 0$, $C_8 > 0$ and $\rho > 0$ sufficiently large

$$
\begin{aligned}
&||Y^\varepsilon - Y^0||^2_{\rho,\mathcal{S}^2(t,T,\mathbb{R}^n)} + \Big(1 - \frac{C_7}{\rho}\Big)||Z^\varepsilon||^2_{\rho,\mathcal{H}^2(t,T,\mathbb{R}^{n\times d})} + \Big(1 - \frac{C_7}{\rho}\Big)\Big|\Big|\frac{V^\varepsilon}{\varepsilon}\Big|\Big|^2_{\mathcal{H}^2_\nu(t,T,\mathbb{R}^n)} \\
&\le C_8\Big(1 + \bar{\mathbb{E}}\Big[\int_t^T \sup_{t\le u\le s} e^{\rho u}|Y_u^\varepsilon - Y_u^0|^2 ds\Big]\Big). \qquad (4.4.10)
\end{aligned}
$$

Gronwall's inequality yields

$$
||X^\varepsilon - X^0||^2_{\mathcal{S}^2(t,T,\mathbb{R}^d)} + ||Y^\varepsilon - Y^0||_{\mathcal{S}^2(t,T,\mathbb{R}^n)} + ||Z^\varepsilon||_{\mathcal{H}^2(t,T,\mathbb{R}^{n\times d})} + \Big|\Big|\frac{V^\varepsilon}{\varepsilon}\Big|\Big|^2_{\mathcal{H}^2_\nu(t,T,\mathbb{R}^n)} \to 0,
$$

as $\varepsilon \to 0$. This finishes the proof of the first claim of the theorem.

2. Analogously to the proof of the first claim, for $\delta, \varepsilon > 0$, if $(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, Z_s^{\varepsilon,\delta}, V_s^{\varepsilon,\delta})_{t\le s\le T}$ solves (4.3.4), it is proved that $(X_s^{\varepsilon,\delta}, Y^{\varepsilon,\delta}, Z_s^{\varepsilon,\delta}, V_s^{\varepsilon,\delta})_{t\le s\le T}$ converges in $\mathcal{M}^2[t,T]$ to $(X_s^{0,\delta}, Y^{0,\delta}, Z_s^{0,\delta}, V_s^{0,\delta})_{t\le s\le T}$, solution of (4.3.4) with $\varepsilon = 0$, as $\varepsilon \to 0$.

Theorem 4.3.2 is still valid if $\varepsilon = 0$. Therefore, the function $u^{0,\delta}(t,x) := Y_t^{t,x,0,\delta}$, $(t,x) \in [T',T] \times \mathbb{R}^d$ given in (4.2.2) is a viscosity solution of the terminal value problem (4.3.1) with $\varepsilon = 0$. Similar conclusion holds when $\delta \to 0$.

For every $\varepsilon, \delta > 0$, the functions $u^{\varepsilon,\delta}$ and $u^{0,\delta}$ are deterministic continuous and uniformly bounded in $\varepsilon, \delta > 0$ (due to **Theorem 4.3.1**) such that, for every $t \in [T',T]$ and $x \in \mathbb{R}^d$ we have

$$
|Y_t^{t,x,\varepsilon,\delta} - Y_t^{t,x,0,\delta}| \le ||Y^{\varepsilon,\delta} - Y^{0,\delta}||_{\mathcal{S}^2(t,T,\mathbb{R}^n)} \to 0 \text{ as } \varepsilon \to 0.
$$

Proposition 4.3.2 implies that there exists $K > 0$ such that for all $(t,x) \in [T',T] \times \mathbb{R}^d$

$$
|u^{\varepsilon,\delta}(t,x) - u^{\varepsilon,\delta}(t',x')| \le K\Big(\mid x - x' \mid^2 + (1 + \mid x \mid^2 \vee |x'|^2) \mid t - t' \mid^2\Big).
$$

Using *Àrzela-Ascoli's theorem* we obtain the uniform convergence of $u^{\varepsilon,\delta}$ to $u^{0,\delta}$ as $\varepsilon \to 0$ in the compact sets of $[T',T] \times \mathbb{R}^d$.

3. Claim 3 follows from analogous arguments.
4. For every $(t, x) \in [T', T] \times \mathbb{R}^d$, we define

$$u(t, x) := Y_t^0,$$

where $(Y_s^0)_{s\in[T',T]}$ is the continuous function that satisfies the backward equation for the deterministic two point boundary value problem (4.4.2).

Since by **Theorem 4.3.1** the family $(u^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $\varepsilon > 0$ and by **Proposition 4.3.2** there exists $K > 0$ such that for every $(t, x) \in [T', T] \times \mathbb{R}^d$,

$$|u^\varepsilon(t, x) - u^\varepsilon(t', x')|^2 \le K\Big(\mid x - x' \mid^2 + K(1+ \mid x \mid^2 \vee |x'|^2) \mid t - t' \mid^2 \Big), \tag{4.4.11}$$

using *Àrzela-Ascoli's theorem* (**Proposition D.5.1**) we derive that there exists a subsequence $(u^{\varepsilon_n})_{n\in\mathbb{N}} \subset (u^\varepsilon)_{\varepsilon>0}$ such that $(u^{\varepsilon_n})_{n\in\mathbb{N}}$ converges uniformly to u in the compact sets of $[T', T] \times \mathbb{R}^d$ as $n \to \infty$. Taking the limit in (4.4.11) we conclude that u is Lipschitz continuous in x and uniformly continuous in t.

Moreover, from **Proposition 4.3.4**, we deduce that u is a viscosity solution in $[T', T] \times \mathbb{R}^d$ of (4.4.3).

Let $v : [T', T] \times \mathbb{R}^d \to \mathbb{R}^k$ be a $C_b^{1,1}([T', T] \times \mathbb{R}^d, \mathbb{R}^n)$ solution of (4.4.3), Lipschitz continuous in x and uniformly continuous in t for (4.4.3). Fixing $(t, x) \in [T', T] \times \mathbb{R}^d$, we define the following function:

$$\psi : [t, T] \to \mathbb{R}^n$$
$$\psi(s) := v(s, X_s).$$

where $(X_s)_{t\le s\le T}$ is given by the solution of the forward equation for the two-point boundary value problem (4.4.2).

Computing its time derivative, for all $j = 1, ..., n$, gives

$$\begin{aligned}
&\frac{d\psi_j}{ds}(s) \\
&= \frac{\partial v_j}{\partial s}(s, X_s^{t,x}) + \sum_{i=1}^{d} \frac{\partial v_j}{\partial x_i}(s, X_s)\frac{\partial (X_s)}{\partial t} \\
&= \frac{\partial v_j}{\partial s}(s, X_s) + \sum_{i=1}^{d} \frac{\partial v}{\partial x_i}(s, X_s) b_i(s, X_s, Y_s) \\
&= -f(s, X_s, v(s, X_s, 0, 0), \\
&\psi(T) = v(T, X_T) = g(x).
\end{aligned}$$

As a consequence, $v(t, x) = v(t, X_t) = u(t, x)$, under the hypothesis that (4.4.2) has a unique solution. So, under this hypothesis, we have (4.4.3) has a unique solution in the space $C_b^{1,1}([T', T] \times \mathbb{R}^d)$. □

4.5 A large deviations principle

In this section we impose furthermore that the measure ν fixed before has an exponentially light density with respect to the Lebesgue measure dz in $\mathbb{R}^d$, i.e.

$$\nu(dz) = e^{-|z|^\alpha} dz, \tag{4.5.1}$$

for some $\alpha > 0$.
We assume this form for the measure ν since we used it in the previous chapters to study the first exit time problem that is described in the Introduction. The main reason is that **Lemma 2.2.1** is used in the sequel of this section. Nevertheless, we could impose less restrictive conditions and replace the exponentially light form of ν with some other integrability conditions. In *Budhiraja et al. (2013)* the reader can find the statement of a large deviations principle for stochastic differential equations in which the Poisson random measure has an intensity satisfying some other integrability conditions than the ones that the measure ν given in (1.1.5) satisfies. We state a large deviations principle for the laws of the forward and the backward process of the FBSDE system (4.2.1) as $\varepsilon \to 0$.
Due to **Theorem 4.2.1**, let $T' < T$ such that, for every arbitrarily fixed $t \in [T'; T]$ we have a unique solution $(X_s^{t,x\varepsilon}, Y^{t,x\varepsilon}, Z_s^{t,x,\varepsilon}, V_s^{t,x,\varepsilon})_{t \le s \le T} \in \mathcal{M}^2[t,T]$ solution of (4.2.1).
For every $\varepsilon > 0$, the key property for the large deviations principle is the decoupling property,

$$Y_s^\varepsilon = u^\varepsilon(s, X_s^\varepsilon), \quad s \in [T', T],$$

that is deduced from **Theorem 4.2.2**. We use a sufficient condition stated in *Budhiraja et al. (2011)* to derive the large deviations principle for laws of the forward process. Using the decoupling property and an extended form of the contraction principle we transfer the large deviations principle of the laws of the forward process to the backward process.
Let us fix $t \in [T', T]$.
We define for every Borel-measurable function $g : [t,T] \times \mathbb{R}^d \longrightarrow [0, \infty)$

$$\mathfrak{L}_T^t(g) := \int_t^T \int_{\mathbb{R}^d} (g(s,z) \ln g(s,z) - g(s,z) + 1) \nu(dz) ds,$$

and for a given square integrable function $\psi \in L^2([t,T], \mathbb{R}^d)$ we define

$$\tilde{\mathfrak{L}}_T^t(\psi) := \frac{1}{2} \int_t^T |\psi(s)|^2 ds.$$

We define $\bar{\mathfrak{L}}_T^t(g, \psi) := \mathfrak{L}_T^t(g) + \tilde{\mathfrak{L}}_T^t(\psi)$.
For every $M > 0$ we consider the sublevel sets of the functionals $\mathfrak{L}_T^t$, $\tilde{\mathfrak{L}}_T^t$ and $\bar{\mathfrak{L}}_T^t$ respectively

by

$$
\begin{aligned}
S_{t,T}^M &:= \Big\{ g : [t,T] \times \mathbb{R}^d \longrightarrow [0,\infty) \mid g \text{ is } \mathcal{B}([t,T]\times\mathbb{R}^d) - \mathcal{B}([0,\infty)) \\
&\qquad \text{measurable such that } \mathfrak{L}_T^t(g) \le M \Big\}, \\
\tilde{S}_{t,T}^M &:= \Big\{ f \in L^2([t,T],\mathbb{R}^d) \mid \tilde{\mathfrak{L}}_T^t(f) \le M \Big\} \text{ and} \\
\bar{S}_{t,T}^M &:= \tilde{S}_{t,T}^M \times S_{t,T}^M.
\end{aligned}
$$

We write $\bar{\mathbb{S}}_{t,T} := \bigcup_{M \ge 0} \bar{S}_{t,T}^M$.

Theorem 4.5.1 (**A large deviations principle**). *We assume that (4.5.1) holds. Under* ***Condition 4.2.2****, by* ***Theorem 4.2.1*** *let $T' < T$, $t \in [T',T]$ and for every $0 \le \varepsilon \le 1$ $(X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon, V_s^\varepsilon)_{t \le s \le T}$ be the unique solution of (4.4.1). We denote, for $x \in \mathbb{R}^d$ and $s \in [t,T]$, $u(s,x) := Y_s^{t,x}$ part of the solution of the following two point boundary value problem for the ODE*

$$
\begin{cases}
\dot{X}_s^{t,x} &= b(s, X_s^{t,x}, Y_s^{t,x}), \\
\dot{Y}_s^{t,x} &= -f(s, X_s^{t,x}, Y_s^{t,x}, 0, 0), \quad t \le s \le T, \\
X_t^{t,x} &= x, \\
Y_T^{t,x} &= g(X_T^{t,x}).
\end{cases}
$$

Then the family $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in the Skorokhod space $\mathbb{D}([t,T],\mathbb{R}^d)$ with good rate function

$$
\begin{aligned}
\mathbb{K} &: C([t,T],\mathbb{R}^d) \longrightarrow [0,\infty], \\
\mathbb{K}(\xi) &:= \inf_{(\varphi,\psi)\in\mathbb{T}_\xi} \bar{\mathfrak{L}}_T^t(\varphi,\psi),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{T}_\xi := \Big\{ (\varphi,\psi) \in \bar{\mathbb{S}}_{t,T} \mid \xi_s = x &+ \int_t^s b(r,\xi_r,u(r,\xi_r))dr + \int_t^s \sigma(r,\xi_r,u(r,\xi_r))\psi_r dr \\
&+ \int_t^s \int_{\mathbb{R}^d} \beta(r,\xi_r)(\varphi(r,z)-1)\nu(dz)ds, \quad s \in [t,T] \Big\},
\end{aligned}
$$

Let us define the nonlinear operator

$$
\begin{aligned}
F &: \mathbb{D}([t,T],\mathbb{R}^d) \longrightarrow \mathbb{D}([t,T],\mathbb{R}^n), \\
F(\xi)(s) &:= u(s,\xi_s), \quad s \in [T,T].
\end{aligned}
$$

Then, the family of backward processes $(Y^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in $\mathbb{D}([t,T],\mathbb{R}^n)$ with good rate function

$$
\begin{aligned}
\mathbb{L} &: C([t,T],\mathbb{R}^n) \longrightarrow [0,\infty], \\
\mathbb{L}(\zeta) &:= \inf\{\mathbb{K}(\xi) \mid F(\xi) = \zeta\}.
\end{aligned}
$$

In order to prove **Theorem 4.5.1** we define the following functional spaces that will be used in the sequel. Consider

$$\bar{\mathcal{A}}^{+}_{t,T} := \Big\{\varphi : [t,T] \times \mathbb{R}^d \times \bar{\mathbb{M}}_{t,T} \longrightarrow [0,\infty) \mid \varphi \text{ is } (\tilde{\mathcal{P}}, \mathcal{B}([0,\infty))) \text{ measurable}\Big\}.$$

As in the preceding chapters, we consider a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ and for every $n \in \mathbb{N}_1$ we define the set of bounded controls

$$\begin{aligned}
\bar{\mathcal{A}}^{+}_{t,T,b,n} &:= \{\varphi \in \bar{\mathcal{A}}^{+}_{t,T} \mid \text{for all } (t,\bar{m}) \in [t,T] \times \bar{\mathbb{M}} : \\
&\qquad \frac{1}{n} \le \varphi(t,x,\bar{m}) \le n, \text{ if } x \in K_n, \text{ and } \quad \varphi(t,x,\bar{m}) = 1, \text{ if } x \in K_n^c\}, \\
\bar{\mathcal{A}}^{+}_{t,T,b} &:= \bigcup_{n\in\mathbb{N}} \bar{\mathcal{A}}^{+}_{t,Tb,n}.
\end{aligned}$$

Let $\mathcal{U}_{t,T} := C([t,T],\mathbb{R}^d) \times \bar{\mathcal{A}}^{+}_{t,T,b} \subset \bar{\mathbb{V}}$ and, for every $M \ge 0$,

$$\bar{\mathcal{U}}^{M}_{t,T} := \Big\{u \in \mathcal{U}_{t,T} \mid u(\omega) \in \bar{S}^{M}_{t,T}\Big\}.$$

For every $M > 0$ and for every $g \in S^M$ we can associate a measure $\nu^g_{t,T} \in \mathbb{M}_{t,T}$, in the following manner,

$$\nu^g_{t,T}(A) := \int_t^T \int_A g(s,z)\nu(dz)ds, \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d).$$

Similarly to what was said in the previous chapters in order to state sufficient conditions for large/moderate deviations principles, this identification turns $S^M_{t,T}$ a compact space, when we consider the topology in $\mathbb{M}_{t,T}$ which convergence is the weak convergence on compact sets of $\mathbb{M}_{t,T}$. This topology is equivalent to the topology which convergence is the vague convergence (see **Definition B.2.2**). In $\tilde{S}^M_{t,T}$ we consider the topology induced by the weak topology of $L^2(t,T,\mathbb{R}^d)$, which makes $\tilde{S}^M_{t,T}$ a compact space. This follows as a direct implication of *Banach-Alaoglu theorem* (**Theorem A.1.1**). We consider the product topology on $\bar{S}^M_{t,T}$ of the two respective topologies described before. Therefore, $\bar{S}^M_{t,T}$ is a compact space.

The following theorem is a sufficient condition for a large deviations principle in the space $\bar{\mathbb{V}}$. We refer the reader to *Budhiraja et al. (2011)*- **Theorem 4.2**.

Theorem 4.5.2 (**Condition for LDP**). *Let $\mathcal{D}$ be a Polish space. For every $\varepsilon > 0$,*

$$\mathcal{G}^\varepsilon : \mathbb{V} \longrightarrow \mathcal{D}$$

and

$$\mathcal{G}^0 : \mathbb{V} \longrightarrow \mathcal{D}$$

are measurable maps satisfying the following conditions.

1. For every $M>0$, let $(f_n,g_n),(f,g)\in \bar{S}^M_{t,T}$ such that $(f_n,g_n)\to(f,g)$ as $n\to\infty$, for the topology described before. Then there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}\subset(g_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{G}^0\Big(\int_t^\cdot f_n(s)ds,\nu^{g_{n_k}}_{t,T}\Big)\to\mathcal{G}^0\Big(\int_t^\cdot f(s)ds,\nu^{g}_{t,T}\Big),$$

as $n\to\infty$, for the topology given by the metric that turns $\mathcal{D}$ into a Polish space.

2. For $M>0$ and $\varepsilon>0$, let $u_\varepsilon=(\varphi_\varepsilon,\psi_\varepsilon),u=(\varphi,\psi)\in\bar{\mathcal{U}}^M_{t,T}$ such that $u_\varepsilon\Rightarrow u$, i.e. in law, as $\varepsilon\to0$. Then we have

$$\mathcal{G}^0\Big(\int_t^\cdot \psi(s)ds,\nu^{\varphi}_{t,T}\Big)\text{ is a limit in law of }\mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}B+\int_t^\cdot\psi_\varepsilon(s)ds,\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}\Big),$$

as $\varepsilon\to0$.

Given $\xi\in\mathcal{D}$, we define the set of the fixed points of ξ by the map $\mathcal{G}^0$,

$$\mathbb{T}_\xi:=\Big\{(f,g)\in\bar{\mathbb{S}}_{t,T}\mid \xi=\mathcal{G}^0\Big(\int_t^\cdot f(s)ds,\nu^g_{t,T}\Big)\Big\}.$$

We define

$$\mathfrak{K}:\mathcal{D}\longrightarrow[0,\infty),$$
$$\mathfrak{K}(\xi):=\inf_{u=(f,g)\in\mathbb{T}_\xi}\bar{\mathfrak{L}}^t_T(u).$$

Under these two conditions, the family of random variables $(Z^\varepsilon)_{\varepsilon>0}$, defined on the probability space $(\bar{\mathbb{V}},\mathcal{B}(\mathbb{V}))$ by

$$Z^\varepsilon:=\mathcal{G}^\varepsilon(\sqrt{\varepsilon}B,\varepsilon N^{\frac{1}{\varepsilon}}),\quad\varepsilon>0$$

satisfies a large deviations principle in $\mathcal{D}$ with good rate function $\mathfrak{K}$.

Proof of Theorem 4.5.1

Proof. For every $0<\varepsilon\le1$, $t\in[T',T]$ and $x\in\mathbb{R}^d$, let $(X^\varepsilon_s,Y^\varepsilon_s,Z^\varepsilon_s,V^\varepsilon_s)_{t\le s\le T}\in\mathcal{M}^2[t,T]$ of

$$\begin{cases}X^\varepsilon_s &= x+\displaystyle\int_t^s b(r,X^\varepsilon_r,Y^\varepsilon_r)dr+\sqrt{\varepsilon}\int_t^s\sigma(r,X^\varepsilon_r,Y^\varepsilon_r)dB_r+\varepsilon\int_t^s\int_{\mathbb{R}^d}\beta(X^\varepsilon_{r-},z)\tilde{N}^{\frac{1}{\varepsilon}}(drdz),\\ Y^\varepsilon_s &= g(X^\varepsilon_T)+\displaystyle\int_s^T f\Big(r,X^\varepsilon_r,Y^\varepsilon_r,Z^\varepsilon_r,\int_{\mathbb{R}^d}V^\varepsilon_r(z)\gamma^\varepsilon(z)\frac{1}{\varepsilon}\nu(dz)\Big)\\ &\quad-\displaystyle\int_s^T Z^\varepsilon_rdB_r-\int_s^T\int_{\mathbb{R}^d}V^\varepsilon_{r-}(z)\tilde{N}^{\frac{1}{\varepsilon}}(drdz),t\le s\le T.\end{cases}\tag{4.5.2}$$

For every $0<\varepsilon\le1$, let $u^\varepsilon(t,x):=Y^\varepsilon_t$ for all $(t,x)\in[T',T]\times\mathbb{R}^d$. As was pointed in **Remark 4.2.4**, Y^ε_t is $\tilde{\mathcal{G}}_t$-measurable. Therefore, the function u^ε is a deterministic function of t and x and due to **Theorem 4.2.2**, for all $s\in[t,T]$,

$$Y^\varepsilon_s=u^\varepsilon(s,X^\varepsilon_s).\tag{4.5.3}$$

The representation of the backward process $(Y_s^\varepsilon)_{t\le s\le T}$ in terms of the forward process $(X_s^\varepsilon)_{t\le s\le T}$ given in (4.5.3) decouples the system of FBSDEs (4.5.2) by, for every $s\in[t,T]$,

$$X_s^\varepsilon = x + \int_t^s b^\varepsilon(r, X_r^\varepsilon)dr + \sqrt{\varepsilon}\int_t^s \sigma^\varepsilon(r, X_r^\varepsilon)dB_r + \varepsilon\int_t^s\int_{\mathbb{R}^D}\beta(X_{r-}^\varepsilon, z)\tilde{N}^{\frac{1}{\varepsilon}}(dr,dz), \quad (4.5.4)$$

where, for every $r\in[t,T]$,

$$\begin{aligned} b^\varepsilon(r, X_r^\varepsilon) &:= b(r, X^\varepsilon, u^\varepsilon(r, X_r^\varepsilon)) \text{ and} \\ \sigma^\varepsilon(r, X_r^\varepsilon) &:= \sigma(r, X^\varepsilon, u^\varepsilon(r, X_r^\varepsilon)). \end{aligned}$$

For every $0<\varepsilon\le 1$, the Lipschitz property of the coefficients of (4.5.2) stated in **Condition 4.2.2**, the Lipschitz continuity of u^ε (**Proposition 4.3.2**) and statement (i) of **Theorem 4.4.1** yield

$$\lim_{\varepsilon\to 0}\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|b^\varepsilon(r, X_r^\varepsilon) - b(r, X_r^0)|^2\Big] = \lim_{\varepsilon\to 0}\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|b^\varepsilon(r, X_r^\varepsilon) - b(r, X_r^0)|^2\Big] = 0,$$

where $(X_r^0, Y_r^0)_{t\le r\le T}$ is the unique continuous solution of the limiting ODE system (4.4.2).

1. We prove the large deviations principle for the laws of the processes $(X^\varepsilon)_{\varepsilon>0}$. For $(t,x)\in[T',T]\times\mathbb{R}^d$, let $u(t,x) := Y_t^{t,x}$, where $(Y_s^{t,x})_{t\le s\le T}$ is the solution of the backward equation of the following two-point boundary value problem

$$\begin{cases} \dot{X}_s^{t,x} &= b(s, X_s^{t,x}, Y_s^{t,x}) \\ \dot{Y}_s^{t,x} &= -f(s, X_s^{t,x}, Y_s^{t,x}, 0, 0), \quad t\le s\le T, \\ X_t^{t,x} &= x, \\ Y_T^{t,x} &= g(X_T^{t,x}). \end{cases}$$

In order to apply **Theorem 4.5.2**, we define the map

$$\mathcal{G}^0 : \mathbb{V} \longrightarrow C([t,T],\mathbb{R}^d)$$

$$\mathcal{G}^0\Big(\int_t^\cdot \psi(s)ds, \nu_{t,T}^\varphi\Big) := \xi,$$

where, for every $s\in[t,T]$,

$$\begin{aligned} \xi_s = x &+ \int_t^s b(r,\xi_r,u(r,\xi_r))dr + \int_t^s \sigma(r,\xi_r,u(r,\xi_r))\psi_r dr \\ &+ \int_t^s\int_{\mathbb{R}^d}\beta(\xi_r,z)(\varphi(r,z)-1)\nu(dz)dr. \end{aligned}$$

In the notation of **Theorem 4.5.2**, for every $\varepsilon>0$, we define the measurable map

$$\mathcal{G}^\varepsilon : \mathbb{V}\longrightarrow \mathbb{D}([t,T],\mathbb{R}^d),$$

such that

$$\mathcal{G}^\varepsilon(f,m)=(\sqrt{\varepsilon}f,\varepsilon m).$$

By definition,

$$\mathcal{G}^\varepsilon(\sqrt{\varepsilon}B,\varepsilon N^{\frac{1}{\varepsilon}}):=X^\varepsilon,$$

where $(X_r^\varepsilon)_{r\in[t,T]}$ satisfies (4.5.4).
We check the conditions of **Theorem 4.5.2**.

i) We prove that, fixed $M>0$, given a sequence $(f_n,g_n)\in\bar{S}_{t,T}^M$ converging to (f,g) in $\bar{S}_{t,T}^M$ as $n\to\infty$ in the topology that was mentioned before and that turns $\bar{S}_{t,T}^M$ into a compact space, it follows that, as $n\to\infty$,

$$\mathcal{G}^0\Big(\int_t^\cdot f_n(s)ds,\nu_{t,T}^{g_n}\Big)\to\mathcal{G}^0\Big(\int_t^\cdot f(s)ds,\nu_{t,T}^{g}\Big),$$

up to a subsequence. For every $n\in\mathbb{N}$, we denote

$$\mathcal{G}^0\Big(\int_t^\cdot f_n(s)ds,\nu_{t,T}^{g_n}\Big)=\xi_n$$
$$\mathcal{G}^0\Big(\int_t^\cdot f(s)ds,\nu_{t,T}^{g}\Big)=\xi,$$

where, for every $s\in[t,T]$

$$\begin{aligned}\xi_n(s)&=x+\int_t^s b(r,\xi_n(r),u(r,\xi_(r)))dr+\int_t^s\sigma(r,\xi_n(r),u(r,\xi_n(r)))f_n(r)dr\\&\quad+\int_t^s\int_{\mathbb{R}^d}\beta(\xi_n(r),z)(g_n(r,z)-1)\nu(dz)dr,\ \text{and}\\\xi(s)&=x+\int_t^s b(r,\xi(r),u(r,\xi(r)))dr+\int_t^s\sigma(r,\xi(r),u(r,\xi(r)))f(r)dr\\&\quad+\int_t^s\int_{\mathbb{R}^d}\beta(\xi_n(r),z)(g(r,z)-1)\nu(dz)dr.\end{aligned}$$

Due to **Proposition A.2.1**, there exists $C>0$ such that

$$\sup_{n\in\mathbb{N}}\sup_{t\le r\le T}|\xi_n(r)|,\ \sup_{t\le r\le T}|\xi(r)|\le C.$$

By the Lipschitz property of b, there exists some constant $K>0$ such that

$$\sup_{n\in\mathbb{N}}\sup_{r\in[t,T]}|b(r,\xi_n(r))|\le K(1+\sup_{n\in\mathbb{N}}\sup_{t\le r\le T}|\xi_n(r)|).$$

For all $t \leq u \leq v \leq T$, the *Cauchy-Schwarz inequality* implies

$$\begin{aligned}
&|\xi_n(u) - \xi_n(v)| \\
&\leq \int_u^v |b(r, \xi_n(r), u(r, \xi_n(r)))|dr + \int_u^v |\sigma(r, \xi_n(r), u(r, \xi_n(r)))||f_n(r)|dr \\
&+ \int_u^v \int_{\mathbb{R}^d} |\beta(r, \xi_n(r))||g_n(r, z) - 1|\nu(dz)dr \\
&\leq K(1 + C)|v - u| + ||\sigma||_\infty ||f_n||_{L^2(t,T,\mathbb{R}^d)} \sqrt{v - u} \\
&+ \int_u^v \int_{\mathbb{R}^d} |\beta(r, \xi_n(r))||g_n(r, z) - 1|\nu(dz)dr. \qquad (4.5.5)
\end{aligned}$$

In view of the assumptions made on β in **Condition 4.2.2** we conclude due to (2.2.3) in **Lemma 2.2.1** that

$$\lim_{\delta \to 0} \sup_{n \in \mathbb{N}} \sup_{|u-v| \leq \delta} \int_u^v \int_{\mathbb{R}^d} |\beta(\xi_n(r), z)||g_n(r, z) - 1|\nu(dz)dr = 0. \qquad (4.5.6)$$

From (4.5.5) and (4.5.6) we have

$$\lim_{\delta \to 0} \sup_{n \in \mathbb{N}} \sup_{|u-v| \leq \delta} |\xi_n(u) - \xi_n(v)| = 0.$$

Hence, $(\xi_n)_{n \in \mathbb{N}}$ is a family of equicontinuous uniformly bounded functions in $C([t, T], \mathbb{R}^d)$. Using *Arzela-Ascoli's theorem*, up to a subsequence, $\xi_n \to \tilde{\xi}$, in the uniform topology in $C([t, T], \mathbb{R}^d)$, for some $\tilde{\xi} \in C([t, T], \mathbb{R}^d)$. Due to the boundedness of σ, the Lipschitz property of b, the Lipschitz continuity of u (**Theorem 4.4.1**) and $\beta \in L^2([t, T] \times \mathbb{R}^d, ds \otimes \nu)$, we may use dominated convergence theorem and pass to the pointwise limit in the equation satisfied by ξ_n, as $n \to \infty$. We conclude that

$$\begin{aligned}
\tilde{\xi}(s) = x &+ \int_t^s b(r, \tilde{\xi}(r), u(r, \tilde{\xi}_r))dr + \int_t^s \sigma(r, \tilde{\xi}_(r), u(r, \tilde{\xi}_r))f(r)dr \\
&+ \int_t^s \int_{\mathbb{R}^d} \beta(\tilde{\xi}(r), z)(g(r, z) - 1)\nu(dz)dr, \quad t \leq s \leq T.
\end{aligned}$$

From the uniqueness of solution for the controlled equation
$\mathcal{G}^0\Big(\int_t^\cdot f(s)ds, \nu_T^g\Big) = \xi$ (see **Proposition A.2.1**) it follows that $\xi = \tilde{\xi}$.

ii) Fix $M > 0$ and for every $\varepsilon > 0$ let $u_\varepsilon = (\varphi_\varepsilon, \psi_\varepsilon), u = (\varphi, \psi) \in \bar{\mathcal{U}}_{t,T}^M$ such that $u_\varepsilon \Rightarrow u$, i.e. in law, as $\varepsilon \to 0$. We prove that

$$\mathcal{G}^0\Big(\int_t^\cdot \psi(s)ds, \nu_{t,T}^\varphi\Big) \text{ is a limit point in law of } \mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}B + \int_t^\cdot \psi_\varepsilon(s)ds, \varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}\Big)$$

as $\varepsilon \to 0$.

For every $\varepsilon > 0$ let $\tilde{\varphi}_\varepsilon := \frac{1}{\varphi_\varepsilon}$. Define the $(\bar{\mathcal{G}}_s)_{t\le s\le T}$-martingales,

$$\mathcal{E}(\psi_\varepsilon)(s) := \exp\Big(\int_t^s \psi_\varepsilon(r)dB_r - \frac{1}{2}\int_t^s |\psi_\varepsilon(r)|^2 dr\Big) \text{ and}$$

$$\tilde{\mathcal{E}}(\tilde{\varphi}_\varepsilon)(s) := \exp\Big(\int_t^s \int_{\mathbb{R}^d}\int_0^{\frac{1}{\varepsilon}} \ln\tilde{\varphi}_\varepsilon(r,z)\bar{N}(ds,dz,dr)$$
$$+ \int_t^s \int_{\mathbb{R}^d}\int_0^{\frac{1}{\varepsilon}} (-\tilde{\varphi}_\varepsilon(r,z)+1)ds\nu(dz)dr\Big).$$

For every $s \in [t,T]$ let $\bar{\mathcal{E}}_s(u_\varepsilon) = \mathcal{E}(\psi_\varepsilon)(s)\tilde{\mathcal{E}}(\varphi_\varepsilon)(s)$. As in the proof of **Theorem 1.2.1** and of **Theorem 1.3.2**, it is immediate that, for every $\varepsilon > 0$ $\Big(\psi_\varepsilon, \frac{1}{\varphi_\varepsilon}\Big)$ satisfies the assumptions of *Girsanov's theorem* (**Theorem B.3.2**) which implies that $(\bar{\mathcal{E}}(u_\varepsilon)(s))_{t\le s\le T}$ is a $(\bar{\mathcal{G}}_s)_{t\le s\le T}$ martingale. Hence, the probability measure defined on $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}))$ by

$$\mathbb{Q}_T^\varepsilon(G) = \int_G \bar{\mathcal{E}}(u_\varepsilon)(T)d\bar{\mathbb{P}}, \qquad \text{for } G \in \mathcal{B}(\bar{\mathbb{V}}),$$

is absolutely continuous with respect to $\bar{\mathbb{P}}$ and under $\mathbb{Q}_T^\varepsilon$ the stochastic process $(\tilde{B}_s^\varepsilon)_{t\le s\le T} := (B_s - \int_t^s \psi_\varepsilon(s)ds)_{t\le s\le T}$ is a Brownian motion and $\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}$ is an independent random measure with the same law of $\varepsilon N^{\frac{1}{\varepsilon}}$ under $\bar{\mathbb{P}}$. We remark the definition of the controlled random measure $N^{\frac{1}{\varepsilon}\varphi_\varepsilon}$,

$$N^{\frac{1}{\varepsilon}\varphi_\varepsilon}((t,s]\times U) = \int_t^s\int_U\int_0^\infty \mathbf{1}_{[0,\frac{1}{\varepsilon}\varphi_\varepsilon]}(r)\bar{N}(dsdzdr).$$

It follows that, for all $0 < \varepsilon \le 1$, $\bar{X}^\varepsilon := \mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}B + \int_t^\cdot \psi_\varepsilon(s)ds, \varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}\Big)$ is the unique strong solution of the following controlled SDE under $\bar{\mathbb{P}}$ (since $\mathbb{Q}^\varepsilon$ is mutually absolutely continuous with $\bar{\mathbb{P}}$):

$$\tilde{X}_s^\varepsilon = x + \int_t^s (b^\varepsilon(r,\tilde{X}_r^\varepsilon) + \sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)\psi_\varepsilon(r))dr + \sqrt{\varepsilon}\int_t^s \sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)dB_r$$
$$+ \int_t^s\int_{\mathbb{R}^d} \beta^\varepsilon(\tilde{X}_r^\varepsilon, z)(\varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz) - \nu(dz)dr), \quad s \in [t,T]. \tag{4.5.7}$$

We do not stress the dependence of the integral on the measure $\mathbb{Q}_T^\varepsilon$.

We write

$$\mathcal{G}^0\Big(\int_t^\cdot \psi, \nu_{t,T}^\varphi\Big) = (\tilde{X}_s)_{t\le s\le T},$$

where $(\tilde{X}_s)_{t\le s\le T}$ is the unique continuous solutions of the controlled ODE, for all $t \le s \le T$,

$$\tilde{X}_s = x + \int_t^s b(r,\tilde{X}_r)dr + \int_t^s \sigma(r,\tilde{X}_r)\psi_r + \int_t^s\int_{\mathbb{R}^d} \beta(\tilde{X}_r,z)(\varphi(r,z)-1)\nu(dz)dr. \tag{4.5.8}$$

We prove in the sequel that we have the convergence in law $\tilde{X}^\varepsilon \Rightarrow \tilde{X}$ as $\varepsilon \to 0$.

a) We prove that there exists some $\varepsilon_0 > 0$ such that the following uniform bound holds,

$$\sup_{0<\varepsilon<\varepsilon_0} \bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T} |\tilde{X}_s^\varepsilon|^2\Big] < \infty. \tag{4.5.9}$$

Applying *Itô's formula* and taking expectations, we have

$$\begin{aligned}
\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T} |\tilde{X}_s^\varepsilon|^2\Big] &\leq |x|^2 + 2\bar{\mathbb{E}}\Big[\int_t^T |\langle b^\varepsilon(r,\tilde{X}_r^\varepsilon), \tilde{X}_r^\varepsilon\rangle| dr\Big] \\
&+ 2\bar{\mathbb{E}}\Big[\int_t^T \langle \sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)\psi_\varepsilon(r), \tilde{X}_r^\varepsilon\rangle dr\Big] \\
&+ 2\bar{\mathbb{E}}\Big[\sup_{t\leq u\leq s}\Big|\int_t^s \langle \tilde{X}_r^\varepsilon, \sqrt{\varepsilon}\sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)d\tilde{B}_r\rangle\Big|\Big] \\
&+ \varepsilon\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\int_t^s |\sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)|^2 dr\Big] \\
&+ 2\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\Big|\int_t^s\int_{\mathbb{R}^d} \langle \tilde{X}_r^\varepsilon, \beta(\tilde{X}_r^\varepsilon,z)\rangle(\varphi_\varepsilon(r,z)-1)\nu(dz)dr\Big|\Big] \\
&+ \bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\Big|\int_t^s\int_{\mathbb{R}^d} |\varepsilon\beta(\tilde{X}_r^\varepsilon,z)|^2 + 2\langle\varepsilon\beta(\tilde{X}_r^\varepsilon,z), \tilde{X}_r^\varepsilon\rangle \tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz)\Big|\Big] \\
&+ \bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d} \varepsilon|\beta(\tilde{X}_r^\varepsilon,z)|^2\varphi_\varepsilon(r,z)\nu(dz)dr\Big].
\end{aligned} \tag{4.5.10}$$

From straightforward estimates we conclude the existence of some constant $C_1 = C_1(||\sigma||_\infty, M, K_1)$, where $K_1 > 0$ is the Lipschitz constant of the vector field b given in **Condition 4.2.2**,

$$\begin{aligned}
&2\bar{\mathbb{E}}\Big[\int_t^T |\langle b^\varepsilon(r,\tilde{X}_r^\varepsilon), \tilde{X}_r^\varepsilon\rangle| dr + \int_t^T |\langle \sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)\psi_\varepsilon(r), \tilde{X}_r^\varepsilon\rangle| dr\Big] \\
&\leq \bar{\mathbb{E}}\Big[\int_t^T (|b^\varepsilon(r,\tilde{X}_r^\varepsilon)|^2 + |\tilde{X}_r^\varepsilon|^2) dr\Big] + ||\sigma||_\infty^2 ||\psi_\varepsilon||_{L^2}^2 \\
&\leq C_1\Big(1 + \bar{\mathbb{E}}\Big[\int_t^T \sup_{t\leq r\leq u} |\tilde{X}_r^\varepsilon|^2 du\Big]\Big).
\end{aligned} \tag{4.5.11}$$

Using *Burkholder-Davis-Gundy's inequalities* (**Proposition B.3.3**) it follows, for some constant $C_2 > 0$,

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\Big|\int_t^s \langle \tilde{X}_r^\varepsilon, \sqrt{\varepsilon}\sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)d\tilde{B}_r\rangle\Big|\Big] \\
&\leq C_2\mathbb{E}\Big[\int_t^T |\langle \tilde{X}_r^\varepsilon, \sqrt{\varepsilon}\sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)\rangle|^2 dr\Big] \\
&\leq C_2\varepsilon||\sigma||_\infty^2 \bar{\mathbb{E}}\Big[\int_t^T \sup_{t\leq r\leq u} |\tilde{X}_r^\varepsilon|^2 du\Big].
\end{aligned} \tag{4.5.12}$$

Trivially we have

$$\varepsilon\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\int_t^s|\sigma^\varepsilon(r,\tilde{X}_r^\varepsilon)|^2dr\Big]\leq\varepsilon||\sigma||_\infty^2(T-t). \tag{4.5.13}$$

Since β is bounded there exists some constant $C_3>0$ such that

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}\Big|\int_t^s\int_{\mathbb{R}^d}\langle\tilde{X}_r^\varepsilon,\beta(\tilde{X}_r^\varepsilon,z)\rangle(\varphi_\varepsilon(r,z)-1)\nu(dz)dr\Big|\Big]\\
&\leq\bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d}|\beta(\tilde{X}_r^\varepsilon,z)||\tilde{X}_r^\varepsilon||\varphi_\varepsilon(r,z)-1|\nu(dz)dr\Big]\\
&\leq\bar{\mathbb{E}}\Big[\int_t^T\sup_{t\leq r\leq u}(1+2|\tilde{X}_r^\varepsilon|^2)\int_{\mathbb{R}^d}|\beta(\tilde{X}_r^\varepsilon,z)||\varphi_\varepsilon(u,z)-1|\nu(dz)du\Big]\\
&\leq C_3\Big(1+\bar{\mathbb{E}}\Big[\int_t^T\sup_{t\leq r\leq u}|\tilde{X}_r^\varepsilon|^2\Big(\int_{\mathbb{R}^d}|\varphi_\varepsilon(u,z)-1|\nu(dz)\Big)du\Big]\Big).
\end{aligned} \tag{4.5.14}$$

We define, for all $s\in[t,T]$,

$$\begin{cases}M_s^1 &:=\int_t^s\int_{\mathbb{R}^d}|\varepsilon\beta(\tilde{X}_{r-}^\varepsilon,z)|^2\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz)\\ M_s^2 &:=\int_t^s\int_{\mathbb{R}^d}2\langle\varepsilon\beta(\tilde{X}_{r-}^\varepsilon,z),\tilde{X}_{r-}^\varepsilon\rangle\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz).\end{cases}$$

With these definitions, we have

$$\begin{aligned}
\bar{\mathbb{E}}\Big[\sup_{t\leq s\leq T}|M_s^1|\Big]&\leq\bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d}|\varepsilon\beta(\tilde{X}_{r-}^\varepsilon,z)|^2N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz)\Big]\\
&\quad+\bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d}|\beta(\tilde{X}_{r-}^\varepsilon,z)|^2\varphi_\varepsilon(r,z)\nu(dz)dr\Big]\\
&\leq\sup_{g\in S_{t,T}^M}2\varepsilon\bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d}g(r,z)\nu(dz)dr\Big],
\end{aligned} \tag{4.5.15}$$

which converges to zero as $\varepsilon\to 0$ since β is bounde.
Due to *Burkholder-Davis-Gundy's inequalities*, we have, for some $C_4>0$, that

may differ from line to line,

$$\begin{aligned}\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|M^2_s|\Big] &\le C_4\bar{\mathbb{E}}\Big[\Big([M^2]_T\Big)^{1/2}\Big]\\ &\le C_4\bar{\mathbb{E}}\Big[\Big(\int_t^T\int_{\mathbb{R}^d}4\varepsilon^2|\langle\beta(\tilde{X}^\varepsilon_{r-},z),\tilde{X}^\varepsilon_r\rangle|^2N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz)\Big)^{1/2}\Big]\\ &\le 2\varepsilon C_4\bar{\mathbb{E}}\Big[\Big(\int_t^T\int_{\mathbb{R}^d}|\beta(\tilde{X}^\varepsilon_{r-},z)|^2|\tilde{X}^\varepsilon_r|^2N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz)\Big)^{1/2}\Big]\\ &\le 2\varepsilon C_4\Big(\bar{\mathbb{E}}\Big[\sup_{t\le r\le T}|\tilde{X}^\varepsilon_r|^2\Big]+\bar{\mathbb{E}}\Big[\int_t^T\int_{\mathbb{R}^d}|\beta(\tilde{X}^\varepsilon_{r-},z)|^2N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz)\Big]\Big)\\ &\le 2\varepsilon C_4\bar{\mathbb{E}}\Big[\sup_{t\le r\le T}|\tilde{X}^\varepsilon_r|^2\Big]+C_3, \end{aligned}\tag{4.5.16}$$

since β is bounded and

$$\begin{aligned}\sup_{g\in S^M_{t,T}}\int_t^T\int_{\mathbb{R}^d}g(r,z)\nu(dz)ds &< \sup_{g\in S^M_{t,T}}\int_t^T\int_{\mathbb{R}^d}\ell(g(r,z))\nu(dz)dr+e\nu(\mathbb{R}^d)(T-t)\\ &\le M+e\nu(\mathbb{R}^d)(T-t).\end{aligned}$$

due to (2.2.14) from **Lemma 2.2.1**.
Combining (4.5.11), (4.5.12), (4.5.13), (4.5.14) with (4.5.10) and using *Gronwall's inequality* (**Proposition A.1.1**), we conclude, for some $C>0$,

$$\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|\tilde{X}^\varepsilon_s|^2\Big]\le C\Big(1+\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|M^1_s|\Big]+\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|M^2_s|\Big]\Big).$$

Finally, using (4.5.15) and (4.5.16), there exists $\tilde{C}>0$ and $\varepsilon_0<\frac{1}{2CC_4}$ such that, for all $\varepsilon<\varepsilon_0$, we have

$$\bar{\mathbb{E}}\Big[\sup_{t\le r\le T}|\tilde{X}^\varepsilon_r|^2(1-2\varepsilon CC_4)\Big]\le\tilde{C}.$$

b) We prove in what follows that the laws of the family of stochastic processes $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ are tight. We write $(\tilde{X}^\varepsilon_s)_{t\le s\le T}$ for the strong solution of the controlled SDE (4.5.7) as

$$\tilde{X}^\varepsilon_s=x+J^\varepsilon_s+\tilde{M}^\varepsilon_s,$$

where, for all $t\le s\le T$,

$$\begin{cases}J^\varepsilon_s &= \displaystyle\int_t^s(b^\varepsilon(r,\tilde{X}^\varepsilon_r)+\sigma^\varepsilon(r,\tilde{X}^\varepsilon_r)\psi_\varepsilon(r)dr+\int_t^s\int_{\mathbb{R}^d}\beta(\tilde{X}^\varepsilon_r,z)(\varphi_\varepsilon(r,z)-1)\nu(dz)dr,\\ \tilde{M}^\varepsilon_s &= \sqrt{\varepsilon}\displaystyle\int_t^s\sigma^\varepsilon(r,\tilde{X}^\varepsilon_r)dB_r+\int_t^s\int_{\mathbb{R}^d}\varepsilon\beta(\tilde{X}^\varepsilon_{r-},z)\tilde{N}^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz).\end{cases}$$

For every $\sigma > 0$ by *Young's inequality* (**Remark A.1.2**) we have

$$\sup_{g \in S_{t,T}^M} \int_t^T \int_{\mathbb{R}^d} g(r,z)\nu(dz)ds \leq M + e\nu(\mathbb{R}^d)(T-t)$$
$$< \infty,$$

since $\nu(\mathbb{R}^d) < \infty$ (**Remark 1.1.5**). The observation above and the boundedness of β imply

$$\begin{aligned}
\bar{\mathbb{E}}\Big[[\tilde{M}^\varepsilon]_T\Big] &= \varepsilon \bar{\mathbb{E}}\Big[\int_t^T |\sigma^\varepsilon(r, \tilde{X}_r^\varepsilon)|^2 dr\Big] + \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} \varepsilon^2 |\beta(\tilde{X}_r^\varepsilon, z)|^2 N^{\frac{1}{\varepsilon}\varphi_\varepsilon}(dr,dz)\Big] \\
&\leq \varepsilon ||\sigma||_\infty^2 (T-t) + \varepsilon \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} |\beta(\tilde{X}_r^\varepsilon, z)|^2 \varphi_\varepsilon(r,z)\nu(dz)dr\Big] \\
&\leq \varepsilon ||\sigma||_\infty^2 (T-t) + \varepsilon \sup_{g \in S_{t,T}^M} \bar{\mathbb{E}}\Big[\int_t^T \int_{\mathbb{R}^d} |\beta(\tilde{X}_r^\varepsilon, z)|^2 g(r,z)\nu(dz)dr\Big] \\
&\leq \varepsilon ||\sigma||_\infty^2 (T-t) + \varepsilon \sup_{g \in S_{t,T}^M} \int_t^T \int_{\mathbb{R}^d} |g(r,z)\nu(dz)dr, \qquad (4.5.17)
\end{aligned}$$

which converges to zero, as $\varepsilon \to 0$.
We prove next that, for every $\tau > 0$, there exists $\varepsilon_0 > 0$ and $\delta = \delta_\tau > 0$ such that,

$$\sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{P}}\Big(\sup_{0<v-u<\delta} |J_v^\varepsilon - J_u^\varepsilon| > \tau\Big) < \tau. \qquad (4.5.18)$$

Straightforward estimates and *Markov-Chebyshev's inequality* yield, for all $\varepsilon < \varepsilon_0$ such that (4.5.9) holds,

$$\begin{aligned}
&\sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{P}}\Big(\sup_{0<v-u<\delta} |J_v^\varepsilon - J_u^\varepsilon| > \tau\Big) \\
&\leq \sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{P}}\Big(\sup_{0<v-u<\delta} \Big|\int_u^v b^\varepsilon(r, \tilde{X}_r^\varepsilon)dr\Big| > \frac{\tau}{3}\Big) \\
&+ \sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{P}}\Big(\sup_{0<v-u<\delta} \Big|\int_u^v \sigma^\varepsilon(r, \tilde{X}_r^\varepsilon)\psi_\varepsilon(r)dr\Big| > \frac{\tau}{3}\Big) \\
&+ \sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{P}}\Big(\sup_{0<v-u<\delta} \Big|\int_u^v \beta(\tilde{X}_r^\varepsilon, z)(\varphi_\varepsilon(r,z) - 1)\nu(dz)dr\Big| > \frac{\tau}{3}\Big) \\
&\leq \frac{9\delta^2}{\tau^2} \sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{E}}\Big[\sup_{t \leq s \leq T} |b^\varepsilon(r, \tilde{X}_r^\varepsilon)|^2\Big] \\
&+ \frac{3}{\tau} \sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{E}}\Big[\sup_{0<v-u<\delta} \int_u^v |\sigma^\varepsilon(r, \tilde{X}_r^\varepsilon)||\psi_\varepsilon(r)|dr\Big] \\
&+ \frac{3}{\tau} \sup_{\varepsilon_0 > \varepsilon > 0} \bar{\mathbb{E}}\Big[\sup_{0<v-u<\delta} \int_u^v |\beta(\tilde{X}_r^\varepsilon, z)||\varphi_\varepsilon(r,z) - 1|\nu(dz)dr\Big]. \qquad (4.5.19)
\end{aligned}$$

Since β is bounded we choose $\delta_0 > 0$ such that, for $\delta < \delta_0$, we have

$$\bar{\mathbb{E}}\Big[\sup_{0<v-u<\delta}\int_u^v\int_{\mathbb{R}^d}|\beta(\tilde{X}_r^\varepsilon,z)||\varphi_\varepsilon(r,z)-1|\nu(dz)dr\Big] < \frac{\tau^2}{9}. \tag{4.5.20}$$

Condition 4.2.2, the Lipschitz continuity of u^ε due to **Proposition 4.3.2** and the uniform bound (4.5.9) imply that there exists some $K > 0$ such that

$$\begin{aligned}\bar{\mathbb{E}}\Big[\sup_{t\le r\le T}|b^\varepsilon(r,\tilde{X}_r^\varepsilon)|\Big] &\le K\Big(1+\sup_{\varepsilon_0>\varepsilon>0}\bar{\mathbb{E}}\Big[\sup_{t\le r\le T}|\tilde{X}_r^\varepsilon|\Big]\Big)\\ &\le K(1+C).\end{aligned} \tag{4.5.21}$$

where, due (4.5.9), the constant $C > 0$ is such that

$$\sup_{\varepsilon_0>\varepsilon>0}\mathbb{E}\Big[\sup_{t\le r\le T}|\tilde{X}_r^\varepsilon|\Big] < C.$$

The fact that σ is bounded and $\varphi_\varepsilon \in \tilde{S}_{t,T}^M$ and the estimates (4.5.19), (4.5.20), (4.5.21) yield

$$\sup_{\varepsilon_0>\varepsilon>0}\bar{\mathbb{P}}\Big(\sup_{0<v-u<\delta}|J_v^\varepsilon - J_u^\varepsilon| > \tau\Big) \le \frac{9\delta^2}{\tau^2}K(1+C) + \frac{3}{\tau}\sqrt{M}||\sigma||_\infty\sqrt{\delta} + \frac{\tau}{3}.$$

For $\delta < \frac{\tau}{3\sqrt{K(1+C)}} \wedge \frac{\tau^2}{9||\sigma||_\infty^2 M}$, we conclude that

$$\sup_{\varepsilon_0>\varepsilon>0}\bar{\mathbb{P}}\Big(\sup_{0<v-u<\delta}|J_v^\varepsilon - J_u^\varepsilon| > \tau\Big) < \tau.$$

This implies, with the same reasoning employed in the proof of **Theorem 1.2.1**, that $(J^\varepsilon)_{\varepsilon>0}$ is C-tight (see **Definition C.2.2**). **Proposition C.2.3** combined with (4.5.17) and (4.5.18) imply that the laws of the family $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ are tight. Using *Prokhorov's theorem* (**Proposition C.1.5**) there exists a weak limit for the laws of $(\tilde{X}^\varepsilon)_{\varepsilon>0}$. Due to *Skorokhod representation theorem* (**Proposition C.1.7**) there exists a random variable $\bar{X}$ such that $\tilde{X}^\varepsilon \to \bar{X}$, $\bar{\mathbb{P}} - a.s.$ Since $\psi_\varepsilon \rightharpoonup \psi$ weakly in $L^2([t,T],\mathbb{R}^d)$ and $\nu_T^{\varphi_\varepsilon} \to \nu_T^\varphi$ in the vague topology of $\times\mathbb{M}_{t,T}$, by the sublinearity of the coefficients of (4.5.7) and the uniform bound on the second moments of $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ stated in (4.5.9), we can take the expectations and pass to the limit pointwise in (4.5.7) and conclude that $(\bar{X}_s)_{t\le s\le T}$ satisfies (4.5.8) $\bar{\mathbb{P}}$-a.s. The uniqueness property for the solution of (4.5.8) stated in **section A.2** in the **Appendix**, implies that $\tilde{X}_s = \bar{X}_s$ for all $t \le s \le T$, $\bar{\mathbb{P}}$-a.s.
Hence, we conclude that

$$\mathcal{G}^0\Big(\int_t^\cdot \psi(s)ds, \nu_{t,T}^\varphi\Big) \text{ is a weak limit in law of } \mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}B + \int_t^\cdot \psi_\varepsilon(s)ds, \varepsilon N^{\frac{1}{\varepsilon}\varphi_\varepsilon}\Big),$$

as $\varepsilon \to 0$.

Theorem 4.5.2 implies that $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in the Skorokhod space $\mathbb{D}([t,T],\mathbb{R}^d)$ with good rate function

$$\mathbb{K}:\mathbb{D}([t,T],\mathbb{R}^d)\longrightarrow[0,\infty],$$

defined by

$$\mathbb{K}(\xi)=\inf_{(\psi,\varphi)\in\mathbb{T}_\xi}\frac{1}{2}\int_t^T|\psi_r|^2dr+\int_t^T(\varphi(r,z)\ln\varphi(r,z)-\varphi(r,z)+1)\nu(dz)dr,$$

where

$$\begin{aligned}\mathbb{T}_\xi=\Big\{&(\psi,\varphi)\in\bar{\mathbb{S}}_{t,T}\mid \text{for all } t\le s\le T\\ &\xi_s=\int_t^s b(r,\xi_r)dr+\int_t^s\sigma(r,\xi_r)\psi_r dr+\int_t^s\int_{\mathbb{R}^d}\beta(\xi_r,z)\nu(dz)dr.\Big\}\end{aligned}$$

2. We transfer the large deviations principle from the laws of $(X_s^\varepsilon)_{t\le s\le T}$ to the laws of the backward process $(Y^\varepsilon)_{\varepsilon>0}$, using an extended form of the contraction principle, stated in **Theorem D.2.2**. We consider the following family of nonlinear operators, indexed in $\varepsilon>0$,
$$\begin{aligned}&F^\varepsilon:\mathbb{D}([t,T],\mathbb{R}^d)\longrightarrow\mathbb{D}([t,T],\mathbb{R}^n)\\ &F^\varepsilon(\xi)(s):=u^\varepsilon(s,\xi_s),\quad s\in[t,T].\end{aligned}$$
We observe that $Y^\varepsilon=F^\varepsilon(X^\varepsilon)$.

i) Fix $\varepsilon>0$. We start to prove the continuity of the map F^ε. Let $y\in\mathbb{D}([t,T],\mathbb{R}^d)$ be fixed arbitrarily and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}([t,T],\mathbb{R}^n)$ converging to y in the *Skorokhod topology* (1.2.3). By definition there exists a sequence of increasing homeomorphisms $\lambda_n:[t,T]\to[t,T]$ such that
$$\begin{aligned}&\lim_{n\to\infty}\sup_{t\le s\le T}|\lambda_n(s)-s|=0 \text{ and}\\ &\lim_{n\to\infty}\sup_{t\le s\le T}|y_n(\lambda_n(s))-y(s)|=0.\end{aligned}\tag{4.5.22}$$
Fix $\delta>0$. Since $d_{J_1}(y_n,y)\to0$ as $n\to\infty$, by **Proposition C.2.2**, let $M>0$ be a uniform bound in $n\in\mathbb{N}$ to $||y_n||_\infty$ and to $||y||_\infty$. Given $r\in[t,T]$, due to **Proposition 4.3.2**, there exists $K>0$ such that
$$\begin{aligned}&|F^\varepsilon(y_n(\lambda_n(r)))-F^\varepsilon(y(\lambda(r)))|^2\\ &=|u^\varepsilon(\lambda_n(r),y_n(\lambda_n(r)))-u^\varepsilon(r,y(r))|^2\\ &\le K(|y_n(\lambda_n(r))-y(r)|^2+(1+|y_n(\lambda_n(r))|^2\vee|y(r)|^2)|\lambda_n(r)-r|^2)),\end{aligned}$$
which converges to zero as $\varepsilon\to0$, due to (4.5.22). This proves that, for every $\varepsilon>0$, $F^\varepsilon(y_n)\to F^\varepsilon(y)$, as $n\to\infty$, in the *Skorokhod topology.*

ii) We next show the convergence on compact sets of $\mathbb{D}([t,T],\mathbb{R}^d)$ of F^ε to F as $\varepsilon \to 0$, where the limit operator is defined by

$$F : \mathbb{D}([t,T],\mathbb{R}^d) \longrightarrow \mathbb{D}([t,T],\mathbb{R}^n),$$

$$F(\eta)(s) := u(s,\eta_s), \quad t \le s \le T.$$

The function $u(t,x) := Y_t^{t,x}$, for $(t,x) \in [0,T] \times \mathbb{R}^d$ is defined by the two point boundary value problem for the ODE

$$\begin{cases} \dot{X}_s^{t,x} &= b(s, X_s^{t,x}, Y_s^{t,x}), \\ \dot{Y}_s^{t,x} &= -f(s, X_s^{t,x}, Y_s^{t,x}, 0, 0), \quad t \le s \le T, \\ X_t^{t,x} &= x, \\ Y_T^{t,x} &= g(X_T^{t,x}). \end{cases}$$

We fix a compact set $K \subset \mathbb{D}([t,T],\mathbb{R}^d)$ for the J_1-topology. We consider the image set

$$A := \Big\{ \xi_s \mid \xi \in K, s \in [t,T] \Big\} \subset \mathbb{R}^d.$$

We prove that A is a compact set in the usual topology of $\mathbb{R}^d$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of elements of A. For every $n \in \mathbb{N}$, there exists $\varphi_n \in K$ and $s_n \in [t,T]$ such that $y_n = \varphi_n(s_n)$. Since $K \subset \mathbb{D}([t,T],\mathbb{R}^d)$ and $[t,T] \subset \mathbb{R}$ are compact sets, there exists $\varphi \in K$ and $s \in [t,T]$ such that $s_n \to s$ and $\varphi_n \to \varphi$ in the *Skorokhod topology*, as $n \to \infty$. By definition of convergence in the J_1-metric, there exists a sequence of increasing homeomorphisms $\lambda_n : [t,T] \to [t,T]$ such that

$$\lim_{n\to\infty} \sup_{t\le s\le T} |\lambda_n(s) - s| = 0 \text{ and}$$
$$\lim_{n\to\infty} \sup_{t\le s\le T} |\varphi_n(\lambda_n(s)) - \varphi(s)| = 0. \tag{4.5.23}$$

We define $y = \varphi(s) \in A$ the candidate for the limit of the sequence $(y_n)_{n\in\mathbb{N}} \subset A$. Fix $\delta > 0$. Due to (4.5.23), there exists $p_1 \in \mathbb{N}$ such that for every $n \ge p_1$

$$|\varphi(s) - \varphi_n(\lambda_n(s_n))| < \frac{\delta}{2}. \tag{4.5.24}$$

Due to right-continuity of φ_n, there exists $\tau > 0$ such that, for every $u \in [t, s_n + \tau) \subset [t,T]$, we have

$$|\varphi_n(\lambda_n(u)) - \varphi_n(s_n)| < \frac{\delta}{2}. \tag{4.5.25}$$

Due to (4.5.23), let $p_2 \in \mathbb{N}$ such that for every $n \ge p_2$ we have $\lambda_n(s_n) - s_n < \tau$. Hence, for $n \ge p_1 \vee p_2$, it follows from (4.5.24) and (4.5.25)

$$|y - y_n| \le |\varphi(s) - \varphi_n(\lambda_n(s_n))| + |\varphi_n(\lambda_n(s_n)) - \varphi_n(s_n)| < \delta,$$

which finishes the proof that A is a compact in $\mathbb{R}^d$. In the previous section, in the proof of **Theorem 4.4.1**, we proved, for every $x \in \mathbb{R}^d$,

$$\bar{\mathbb{E}}\Big[\sup_{t\le s\le T}|Y_s^{t,x,\varepsilon}-Y_s^{t,x}|^2\Big]\to 0, \text{as } \varepsilon\to 0. \tag{4.5.26}$$

Hence,

$$\begin{aligned}\sup_{\varphi\in K}||F^\varepsilon(\varphi)-F(\varphi)||_\infty^2 &= \sup_{\varphi\in K}\sup_{t\le s\le T}|u^\varepsilon(s,\varphi_s)-u(s,\varphi_s)|^2\\ &\le \sup_{x\in A}\sup_{t\le s\le T}|Y_s^{s,x,\varepsilon}-Y_s^{s,x}|^2\\ &\to 0 \text{ as } \varepsilon\to 0,\end{aligned}$$

which shows the uniform convergence of F^ε to F in the compact sets of $\mathbb{D}([t,T],\mathbb{R}^d)$ and therefore the convergence in the *Skorokhod metric* J_1, since the J_1-topology is finer than the uniform topology in $\mathbb{D}([t,T],\mathbb{R}^d)$.
We are in a position to apply **Theorem D.2.2** and conclude that the family of laws $(\mathbb{P}\circ(Y^\varepsilon)^{-1})_{\varepsilon>0}$ sastifies a large deviations principle in $\mathbb{D}([t,T],\mathbb{R}^n)$ with the good rate function

$$\mathbb{L}(\zeta) := \inf_{F(\varphi)=\zeta,\varphi\in\mathbb{D}([t,T],\mathbb{R}^d)}\mathbb{K}(\varphi), \quad \zeta\in\mathbb{D}([t,T],\mathbb{R}^n).$$

□

Remark 4.5.1. *The asymptotic study and a large deviations principle for a fully coupled FBSDE with jumps is not considered in our study. To our knowledge the asymptotic study and a large deviations statement for a fully coupled FBSDE is not available in the literature. The author believes that this is an interesting topic for future research that requires a deeper study of of the gradient estimates of the function u^ε, solution of the PIDE associated to the FBSDE system. We refer the reader to the thesis of Fromm (2014) for the study of decoupling fields for fully coupled Brownian FBSDEs.*

Appendix

Appendix A

Basic facts used along the text

A.1 Auxiliary results

We list a collection of classical results that are used often along the text. The results listed here can be found in classical textbooks of Analysis, such as in *Brezis (2011)*, *Rudin (1966)* and *Rockafellar (1996)*.

Proposition A.1.1 (Gronwall's inequality). *Let $u, v : [0,T] \longrightarrow \mathbb{R}$ be real valued continuous functions. Assume u is differentiable in $[0,T]$ and satisfies*

$$u'(t) \leq v(t)u(t), \quad t \in [0,T].$$

Then

$$u(t) \leq u(0) \exp \int_0^t v(s)ds, \quad t \in [0,T].$$

Proposition A.1.2 (Backward Gronwall's inequality). *Let $u, v : [0,T] \longrightarrow \mathbb{R}$ be real valued continuous functions and $c \geq 0$. We assume $u, v \geq 0$, u bounded measurable and v a integrable function. If u satisfies*

$$u(t) \leq c + \int_t^T u(s)v(s)ds, \quad t \in [0,T],$$

then

$$u(t) \leq u(0) \exp \int_t^T v(s)ds, \quad t \in [0,T].$$

Proof. We assume without loss of generality $c > 0$. Let us define

$$z(t) = c + \int_t^T u(s)v(s)ds, \text{ for } 0 \leq t \leq T.$$

It follows that

$$\ln z(T) - \ln z(t) = -\int_t^T \frac{u(s)v(s)}{z(s)} ds.$$

Since $u \leq z$ in $[0,T]$,

$$\ln z(t) \leq \ln c + \int_t^T v(s)ds, \qquad \text{for all } t \in [0,T]$$

and the result follows. $\square$

Let $(E, ||.||_E)$ be a Banach space with norm $||.||_E$. We write E^* for the topological dual space . On E^* we consider the weak * topology, which is the coarsest topology such that the functional

$$\varphi \mapsto \langle \varphi, x\rangle_E := \varphi(x), \text{ is continuous, for all } x \in E,$$

where $\langle ., .\rangle_E$ is the dual pairing between E and E^*. A sequence $(\varphi_n)_{n\in\mathbb{N}} \subset E^*$ converges in the weak * topology to $\varphi \in E^*$ if and only if

$$\langle \varphi_n, f\rangle_E \to \langle \varphi, f\rangle_E, \quad \text{for all } f \in E.$$

We write $\varphi_n \rightharpoonup \varphi$.

Theorem A.1.1 (**Banach-Alaoglu theorem**). *The closed unit ball of E^* is compact with respect to the weak* topology.*

As a direct consequence we have the following remark that we use to state a moderate deviations principle in **Chapter 3**, applied to the Hilbert space $L^2([0,T] \times \mathbb{R}^d, \nu_T)$.

Remark A.1.1. *In a Hilbert space (since it is a reflexive space) every bounded and closed set is weakly* relatively compact, hence every bounded sequence has a weakly convergent subsequence.*

Given a function $f : E \longrightarrow (-\infty, \infty]$, the *Fenchel-Legendre conjugate* function of f is the function defined by

$$f^* : Dom(f^*) \subset E^* \longrightarrow [-\infty, \infty]$$
$$f^*(x^*) := \sup_{x\in E}\{\langle x^*, x\rangle_E - f(x)\}, \quad \text{for all } x^* \in E^*,$$

where

$$Dom(f^*) := \Big\{x^* \in E^* \mid \sup_{x\in E}\{\langle x^*, x\rangle_E - f(x)\} < \infty.\Big\}.$$

The next inequality, know as *Young's inequality*, is used often during this text.

Theorem A.1.2 (Young-Legendre's inequality). *Given $f : E \longrightarrow \mathbb{R}$ convex, for every $u \in Dom(f^*), v \in E$ we have*

$$\langle u, v\rangle_E \leq f^*(u) + f(v).$$

As a direct consequence we have the following forms of *Young's inequality.*

Remark A.1.2.

i) Given $f(a) = \frac{a^p}{p}, \quad a \geq 0, p > 0$, the Fenchel-Legendre transform of f is

$$g(b) = \frac{b^q}{q}, \quad \textit{for all } b \geq 0, q \in \mathbb{R}^+ \textit{ such that } \frac{1}{p} + \frac{1}{q} = 1.$$

*Hence, **Young-Legendre's inequality** reads as*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \textit{for all } a, b \geq 0,\ \frac{1}{p} + \frac{1}{q} = 1.$$

ii) For the convex function $f(x) = e^x - 1, \quad x \geq 0$, the convex conjugate of f given by the Fenchel-Legendre's transform is the function

$$g(b) = 1 - b + b\ln b = \ell(b), \quad b \geq 0.$$

Young-Legendre's inequality reads

$$ab \leq e^a + b\ln b - b, \quad a, b \geq 0.$$

Proposition A.1.3 (Arzelà-Ascoli theorem). *Given a sequence $(\varphi_k)_{k\in\mathbb{N}}$, $\varphi_k : [a, b] \to \mathbb{R}^d$ of continuous functions, satisfying the following conditions.*

i) Uniform boundedness; i.e.

$$\sup_{n\in\mathbb{N}} \sup_{a\leq t\leq b} |\varphi_k(t)| < \infty \textit{ and}$$

ii) Equicontinuity; i.e. for every δ there exists $\tau > 0$ such that

$$\sup_{\substack{|t-s|<\tau \\ t,s\in[a,b]}} \sup_{n\in\mathbb{N}} |\varphi_k(t) - \varphi_k(s)| < \delta.$$

Then there exists a subsequence of $(\varphi_k)_{k\in\mathbb{N}}$ that converges uniformly to a continuous function on $[a, b]$.

We use the following more general version of *Arzelà-Ascoli theorem.*

Proposition A.1.4. *Let X be a compact Hausdorff space. Then a subset F of $C(X)$, the space of the continuous functions with values in $\mathbb{R}$, is relatively compact in the topology induced by the uniform norm if and only if it is equicontinuous and pointwise bounded.*

A.2 Controlled ODEs

Proof of Proposition 1.1.2

Proof. Fixed $T > 0$, $x \in \mathbb{R}^d$ and $g \in \mathbb{S}$, under **Condition 1.1.1** we show that there exists a unique $\tilde{X}^g \in C([0,T], \mathbb{R}^d)$ such that

$$\tilde{X}^g_t = x - \int_0^t \nabla U(\tilde{X}^g_s) ds + \int_0^t \int_{\mathbb{R}^d} z(g(s,z) - 1)\nu(dz)ds.$$

Furthermore, we have the uniform bound, for every $M \geq 0$,

$$\sup_{g \in S^M} \sup_{t \in [0,T]} |X^g_t| < \infty.$$

Fix the space

$$\mathbb{X} := \Big\{ u \in C([0,T], \mathbb{R}^d) \mid \sup_{t \in [0,T]} e^{-kt}|u(t)| < \infty \Big\},$$

where $k > 0$ is defined later. $(\mathbb{X}, ||.||_{\mathbb{X}})$ is a Banach space with the norm

$$||u||_{\mathbb{X}} := \sup_{t \in [0,T]} e^{-kt}|u(t)|.$$

We define the solution map

$$\Theta : \mathbb{X} \longrightarrow \mathbb{X},$$

$$(\Theta u)(t) = x - \int_0^t \nabla U(u(s)) ds + \int_0^t \int_{\mathbb{R}^d} z(g(s,z) - 1)\nu(dz)ds, \quad \text{for all } t \geq 0.$$

We note that Θ is well defined, i.e. if $u \in \mathbb{X}$ then $\Theta u \in \mathbb{X}$. We prove this statement in what follows. Given $u \in \mathbb{X}$, let $R > 0$ such that

$$\sup_{t \in [0,T]} |u(t)| < R.$$

Then due to the local growth conditions of ∇U (U is C^2) there exists $K = K(R) > 0$ such that

$$|\nabla U(u(s))| \leq K.$$

Using **Lemma 2.2.1** it follows

$$\int_0^T \int_{\mathbb{R}^d} |z||g(s,z) - 1|\nu(dz)ds < \infty$$

and from the previous statement,

$$\int_0^T |\nabla U(u(s))| ds < \infty,$$

which implies that $\Theta u \in \mathbb{X}$.

1. Let us suppose in a first step that ∇U is globally Lipschitz in $x \in \mathbb{R}^d$, i.e. there exist $K > 0$ such that

$$|\nabla U(x) - \nabla U(y)| \leq K|x - y|, \quad x, y \in \mathbb{R}^d.$$

Given $u, v \in \mathbb{X}$ and $k > 0$

$$e^{-kt}|\Theta u(t) - \Theta v(t)| \leq K \int_0^t e^{-k(t-s)} e^{-ks}|u(s) - v(s)|ds.$$

Therefore,

$$||\Theta u - \Theta v||_{\mathbb{X}} \leq K||u - v||_{\mathbb{X}} \frac{1 - e^{-kT}}{k}$$

If we choose

$$k > K$$

Θ is a contraction on $\mathbb{X}$. Hence, *Banach's fixed point theorem* asserts the existence of a unique element $u \in \mathbb{X}$ such that, for all $t \in [0, T]$,

$$u(t) = x - \int_0^t \nabla U(u(s))ds + \int_0^t \int_{\mathbb{R}^d} z(g(s, z) - 1)\nu(dz)ds.$$

2. Let us consider now, for every $k \in \mathbb{N}$, ∇U_k a globally Lipschitz truncation of ∇U in the variable $x \in \mathbb{R}^d$, such that

$$\nabla U_k(x) = \nabla U(x), \quad \text{for } |x| \leq k$$

According to step 1, for every $k \in \mathbb{N}$ there exists a unique $u_k \in \mathbb{X}$ such that, for all $t \in [0, T]$,

$$u_k(t) = x - \int_0^t \nabla U_k(u_k(s))ds + \int_0^t \int_{\mathbb{R}^d} z(g(s, z) - 1)\nu(dz)ds.$$

It follows for every $t \in [0, T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_k(t)|^2 &= \langle u_k(t), \dot{u}_k(t)\rangle \\ &\leq -\eta|u_k(t)|^2 + |u_k(t)|^2 \int_{\mathbb{R}^d} |g(t, z) - 1|\nu(dz) + \int_{\mathbb{R}^d} |z|^2|g(t, z) - 1|\nu(dz).\end{aligned}$$

Due to the statements (2.2.1) and (2.2.2) of **Lemma 2.2.1**, *Gronwall's inequality* implies

$$\sup_{k \in \mathbb{N}} \sup_{t \in [0,T]} |u_k(t)|^2 < \infty. \tag{A.2.1}$$

Furthermore, the limit (2.2.3) yields

$$\lim_{\delta\to 0}\sup_{k\in\mathbb{N}}\sup_{|t-s|<\delta}|u_k(t)-u_k(s)|=0.$$

We conclude that $(u_k)_{k\in\mathbb{N}}$ is an equicontinuous sequence of bounded functions of $C([0,T],\mathbb{R}^d)$. Àrzela-Ascoli theorem asserts the existence of a subsequence $(u_{k_i})_{i\in\mathbb{N}}\subset(u_k)_{k\in\mathbb{N}}$ that converges uniformly to some $\tilde{X}^g\in C([0,T],\mathbb{R}^d)$. For sake of simplicity we write indistinctively the subsequence $(u_{k_i})_{i\in\mathbb{N}}$ from the sequence $(u_k)_{k\in\mathbb{N}}$. As a consequence we conclude the uniform bound

$$\sup_{g\in S^M}\sup_{t\in[0,T]}|\tilde{X}^g(t)|<\infty.$$

Due to the uniform bound that was derived for the sequence $(u_k)_{k\in\mathbb{N}}$ and by the Lipschitz property of the coefficients ∇U_k, using dominated convergence in the followig expression, for all $t\in[0,T]$,

$$u_k(t)=x-\int_0^t\nabla U_k(u_k(s))ds+\int_0^t\int_{\mathbb{R}^d}z(g(s,z)-1)\nu(dz)ds,$$

we conclude that $\tilde{X}^g$ satisfies the integral equation, for all $t\in[0,T]$,

$$\tilde{X}^g_t=x-\int_0^t\nabla U(\tilde{X}^g_s)ds+\int_0^t\int_{\mathbb{R}^d}z(g(s,z)-1)\nu(dz)ds.$$

3. Uniqueness of solution. Let $\tilde{X}^g$ and $\bar{X}^g$ be two solutions of the integral equation above. Hence, for all $t\in[0,T]$

$$\frac{d}{dt}|\tilde{X}^g(t)-\bar{X}^g(t)|^2\leq-\eta|\tilde{X}^g(t)-\bar{X}^g(t)|^2.$$

Gronwall's inequality implies

$$\sup_{t\in[0,T]}|\tilde{X}^g(t)-\bar{X}^g(t)|^2\leq 0.$$

This concludes the proof.

□

Analogously we conclude the following result.

Proposition A.2.1. *Fix $T>0$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $M>0$, $f\in\tilde{S}^M_{t,T}$ and $g\in S^M_{t,T}$. Consider the following measurable functions*

$$\tilde{b}:[t,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$$
$$\tilde{\sigma}:[t,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}^{d\times d}$$
$$\tilde{\beta}:\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$$

and assume that they satisfy global Lipschitz and dissipativity conditions. Then there exists a unique $\tilde{X} = \tilde{X}^{f,g} \in C([t,T], \mathbb{R}^d)$ *satisfying, for all* $s \in [t,T]$,

$$\tilde{X}_s = x + \int_t^s \tilde{b}(r, \tilde{X}_r)dr + \int_t^s \tilde{\sigma}(r, \tilde{X}_r) f_r dr + \int_t^s \int_{\mathbb{R}^d} \beta(\tilde{X}_r, z)(g(r,z) - 1)\nu(dz)dr.$$

Furthermore, we have the uniform bound

$$\sup_{f \in \tilde{S}^M_{t,T}} \sup_{g \in S^M_{t,T}} \sup_{t \le r \le T} |\tilde{X}_r| < \infty.$$

A.3 About the measure $\nu(dz) = e^{-|z|^\alpha} dz, \quad \alpha > 0$

Proposition A.3.1. *Let* $\nu(dz) = e^{-|z|^\alpha} dz$ *for some* $\alpha > 0$ *where* dz *stands for the Lebesgue measure defined on the Borel sets of* $\mathbb{R}^d$*. The measure* ν *is finite,*

$$\nu(\mathbb{R}^d) < \infty.$$

Proof. We take the generalized spherical coordinates change of variables in $\mathbb{R}^d$,

$$\begin{cases} z_1 = r \prod_{k=1}^{d-1} \sin\theta_k \\ z_i = r\cos\theta_i \prod_{k=i}^{d-1} \sin\theta_k, \quad 2 \le i \le d-1 \\ \cdots\cdots \\ z_d = r\cos\theta_{d-1}, \end{cases}$$

where $r = \sqrt{z_1^2 + \ldots z_d^2}$ and $\theta_1 \in [0, 2\pi)$ and $\theta_i \in [0,\pi)$, $i \in \{2, ..., d-1\}$. The Jacobian of this transformation is (see *Blumenson (1960)*)

$$J = (-1)^{d-1} r^{d-1} \prod_{k=2}^{d-1} \sin^{k-1}\theta_k,$$

which implies

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-|z|^\alpha} dz \\
&= \int_0^\infty \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \Big|(-1)^{d-1} r^{d-1} \prod_{k=2}^{d-1} \sin^{k-1}\theta_k \Big| e^{-r^\alpha} dr d\theta_{d-1} \ldots d\theta_1 \\
&= \Big(\int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \Big|(-1)^{d-1} \prod_{k=2}^{d-1} \sin^{k-1}\theta_k \Big| d\theta_{d-1} \ldots d\theta_1 \Big) \int_0^\infty r^{d-1} e^{-r^\alpha} dr \\
&= \frac{c_d}{\alpha} \int_0^\infty t^{\frac{d-1}{\alpha}} t^{\frac{1-\alpha}{\alpha}} e^{-t} dt \\
&= \frac{c_d}{\alpha} \Gamma\Big(\frac{d}{\alpha}\Big) \\
&< \infty,
\end{aligned}$$

where $c_d = \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} \left| (-1)^{d-1} \prod_{k=2}^{d-1} \sin^{k-1} \theta_k \right| d\theta_{d-1} \dots d\theta_1$. $\square$

Appendix B

Lévy processes and Poisson random measures

We list a collection of definitions and classical results that can be found in any textbook or monography on Lévy processes. We refer to *Applebaum (2009)*, *Bertoin (1998)*, *Kyprianou (2014)*, *Protter (2005)*, *Kunita (2004)* and *Sato (2013)* where all the results and definitions presented in this section are treated.

B.1 Lévy processes

Definition B.1.1 (Lévy process). *A process $L = (L_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a* ***Lévy process*** *if the following conditions are fulfilled:*

1. *L starts at 0 $\mathbb{P}-$a.s., i.e. $\mathbb{P}(L_0 = 0) = 1$;*

2. *L has independent increments, i.e. for $k \in \mathbb{N}$ and $0 \leq t_0 < \cdots < t_k$,*

$$L_{t_1} - L_{t_0}, \ldots, L_{t_k} - L_{t_{k-1}} \quad \textit{are independent};$$

3. *L has stationary increments, i.e., for $0 \leq s \leq t$, $L_t - L_s =^d L_{t-s}$;*

4. *L is stochastically continuous, i.e. for all $t \geq 0$ and $\varepsilon > 0$*

$$\lim_{s\to t} \mathbb{P}(|L_t - L_s| > \varepsilon) = 0.$$

The reader can find the proof of the following result in *Applebaum (2009)*- **Theorem 2.1.8**

Proposition B.1.1. *Every Lévy process has a càdlàg modification that is itself a Lévy process.*

Due to this fact, we assume moreover that every Lévy process has almost surely càdlàg paths.

The Lévy-Khintchine formula is a central classical result that characterizes the law of a Lévy process. More generally it characterizes the law of an infinitely divisible distribution, a class of random variables to which Lévy processes belong. This is the content of the following definition.

Definition B.1.2 (Infinititely divisible distribution). *A random variable X, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$, is said to have an infinitely divisible distribution if, for every $n \in \mathbb{N}$, there exists a sequence of i.i.d. random variables $X_{1,n}, \ldots, X_{n,n}$ defined on the same probability space, such that*

$$X =^d X_{1,n} + \ldots X_{n,n}.$$

Remark B.1.1 (Every Lévy process has an infinitely divisible distribution). *Fix $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathbb{E}$ be the integral of $\mathbb{P}$. The law of a random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ is characterized via its characteristic function. Let $\psi(u) := -\ln \mathbb{E}[e^{i\langle u, X\rangle}]$, for all $u \in \mathbb{R}^d$, be the characteristic exponent of X. Hence, X has an infinitely divisible distribution if, for every $n \in \mathbb{N}$, there exists a characteristic exponent of a probability law, ψ_n such that*

$$\psi(u) = n\psi_n(u), \quad \textit{for all } u \in \mathbb{R}^d.$$

Using the definition it is immediate that a Lévy process has an infinitely divisible distribution. Let $(L_t)_{t\geq 0}$ be a Lévy process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$. For every $n \in \mathbb{N}$, we use a telescopic sum development

$$L_t = L_{\frac{t}{n}} + (L_{\frac{2t}{n}} - L_{\frac{t}{n}}) + \cdots + (L_t - L_{\frac{(n-1)t}{n}}).$$

By the fact that $(L_t)_{t\geq 0}$ has independent and stationary increments this proves that $(L_t)_{t\geq 0}$ has an infinitely divisible distribution. Defining, for all $t \geq 0$ and for every $u \in \mathbb{R}^d$,

$$\psi_t(u) = -\ln \mathbb{E}[e^{i\langle u, L_t\rangle}],$$

from the telescopic sum above, for any $m, n \in \mathbb{N}$,

$$m\psi_1 = \psi_m = n\psi_{\frac{m}{n}},$$

and consequently for any rational $t \in \mathbb{Q}^+$,

$$\psi_t(u) = t\psi_1(u).$$

For any $t \in \mathbb{R}^+ - \mathbb{Q}$, the identity above follows by approximating t by a decreasing sequence of rationals and using dominated convergence theorem. It follows that, for every $t \geq 0$,

$$\mathbb{E}[e^{i\langle u, L_t\rangle}] = e^{-t\psi(u)},$$

where $\psi(u) = \psi_1(u)$, for every $u \in \mathbb{R}^d$. We call ψ the characteristic exponent of $(L_t)_{t\geq 0}$.

In *Applebaum (2009)*- **Theorem 1.2.14** it is proved the following result that characterizes the law of an infinitely divisible distribution and therefore the law of a Lévy process.

Theorem B.1.1 (Lévy- Khintchine formula). *A probability law μ of a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ is infinitely divisible with characteristic exponent ψ, defined by*

$$\int_{\mathbb{R}^d} e^{i\langle u,z\rangle}\mu(dz) = e^{-\psi(u)}, \quad \text{for all } u \in \mathbb{R}^d,$$

if and only if there exists a triple, called ***Lévy triplet****, (b, σ, ν), where $b \in \mathbb{R}^d$, σ is a positive-definite symmetric matrix of $\mathbb{R}^{d\times d}$ and ν is a measure defined on the Borel sets of $\mathbb{R}^d$, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(dz) < \infty$ such that, for every $u \in \mathbb{R}^d$,*

$$\psi(u) = i\langle b, u\rangle + \frac{1}{2}\langle u, \sigma u\rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle u,z\rangle} + i\langle u, z\rangle 1_{\{|z|<1\}})\nu(dz).$$

The measure ν is called the ***Lévy measure*** *of the infinitely divisible distribution.*

The most popular Lévy process is **Brownian motion**.

Example B.1.1 (Brownian motion). *A* ***Wiener process*** *(or* ***Brownian motion****) $(B_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ is a stochastic process satisfying the following conditions:*

1. $\mathbb{P}(B_0 = 0) = 1$;

2. *It is a process with independent and stationary increments;*

3. $B_t \sim$ *Gaussian*$(0, t)$, *for every* $t \geq 0$;

4. *The paths of $(B_t)_{t\geq 0}$, $\omega \mapsto B_t(\omega)$ are continuous $\mathbb{P}$-a.s.*
   Given $\sigma \in \mathbb{R}^{d\times d}$ symmetric positive definite $(B_t)_{t\geq 0}$ is a Wiener process with covariance σ if it satisfies all the properties above but with the condition that, for every $t \geq 0$, B_t is a Gaussian process with mean zero and covariance σ. Then its characteristic exponent is given by

$$\psi_t(u) = \frac{1}{2}\langle u, \sigma u\rangle t, \quad t \geq 0, u \in \mathbb{R}^d.$$

$(B_t)_{t\geq 0}$ is the only Lévy process that is a continuous martingale. This can be seen using Lévy's martingale representation of Brownian motion, for example in Applebaum (2009)- ***Theorem 2.2.7****.*

Another important stochastic process in the class of Lévy processes is the **Poisson process**.

Example B.1.2 (Poisson process). *A stochastic process $P = (P_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in the non-negative integers is called a* ***Poisson process*** *with intensity $\lambda > 0$ if:*

1. *the paths of P are $\mathbb{P}$-a.s. right-continuous with left-limits (càdlàg);*
2. $\mathbb{P}(P_0 = 0) = 1$;
3. *P has stationary increments;*
4. *for all $0 \leq s \leq t$ $P_t - P_s$ is independent of $\{P_u \mid u \leq s\}$;*
5. *for all $t > 0$, $P_t \sim Poisson(\lambda t)$.*

From straight-forward computations,

$$\begin{aligned}
\mathbb{E}[e^{\langle iu, N_t\rangle}] &= \sum_{k\geq 0} e^{iku}\mathbb{P}(P_t = k) \\
&= \sum_{k\geq 0} e^{iku} e^{-\lambda}\frac{\lambda^k}{k!} \\
&= e^{-\lambda(1-e^{iu})} \\
&= \left(e^{-\frac{\lambda}{n}(1-e^{iu})}\right)^n,
\end{aligned}$$

which shows that $(P_t)_{t\geq 0}$ has an infinitely divisible law. The characteristic exponent is, for all $u \in \mathbb{R}^d$,

$$\psi(u) = \lambda(1 - e^{iu}).$$

The corresponding Lévy triplet of $(P_t)_{t\geq 0}$ is $(0, 0, \nu)$, where $\nu = \lambda\delta_1$, with δ_1 the Dirac measure in 1.

In this thesis we study the asymptotic first exit time of a perturbed dynamical system by a specific Lévy process, which is a **compensated compound Poisson process**.

Example B.1.3 (Compound Poisson process). *We suppose that $(P_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ and $(\xi_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables independent of $(P_t)_{t\geq 0}$, with values in $\mathbb{R}^d$ and with non-atomic law μ at 0. We construct the following stochastic process*

$$X_t = \sum_{i=0}^{P_t} \xi_i,$$

*We call $(X_t)_{t\geq 0}$ a **compound Poisson Process**. Computing the characteristic function of $(P_t)_{t\geq 0}$ we obtain, for all $u \in \mathbb{R}^d$ and $t \geq 0$,*

$$\begin{aligned}\mathbb{E}[e^{i\langle u, X_t\rangle}] &= \sum_{n\geq 0} \mathbb{E}[e^{i\langle u, \sum_{i=1}^n \xi_i\rangle}]\mathbb{P}(P_t = n)\\ &= \sum_{n\geq 0} \Big(\int_{\mathbb{R}^d} e^{i\langle u,z\rangle}\mu(dz)\Big)^n e^{-\lambda}\frac{\lambda^n}{n!}\\ &= e^{-\lambda \int_{\mathbb{R}^d}(1-e^{i\langle u,z\rangle})\mu(dz)}.\end{aligned}$$

Using the Lévy-Khintchine formula we see that the Lévy triplet of $(X_t)_{t\geq 0}$ is $(b, 0, \nu)$ with

$$\begin{aligned}b &= -\lambda \int_{0<|z|<1} z\mu(dz), \ \ \text{and}\\ \nu(dz) &= \lambda\mu(dz).\end{aligned}$$

Conversely, given a pure jump Lévy process $(L_t)_{t\geq 0}$ with Lévy triplet $(0, 0, \nu)$ and ν finite intensity, $(L_t)_{t\geq 0}$ is a jump process that has only a finite number of jumps in every bounded time interval. We can associate a Poisson process $(P_t)_{t\geq 0}$ and a sequence of i.i.d random variables $(\xi_i)_{i\in\mathbb{N}}$ with values in $\mathbb{R}^d$ independent of $(P_t)_{t\geq 0}$ such that

$$L_t = \sum_{i=1}^{P_t} \xi_i.$$

The intensity of $(P_t)_{t\geq 0}$ is $\nu(\mathbb{R}^d)$ and the law of the i.i.d. sequence $(\xi_i)_{i\in\mathbb{N}}$ is $\frac{\nu}{\nu(\mathbb{R}^d)}$. Furthermore, the waiting times between two sucessive jumps are also independent and identically distributed exponential random variables. A compound Poisson process can be seen as a random walk whose jumps have been zoomed out in identically independent distributed periods of time.
Associated to $(L_t)_{t\geq 0}$ is $(\tilde{L}_t)_{\varepsilon>0}$ the compensated compound Poisson process which is defined, for all $t \geq 0$, by

$$\tilde{L}_t = L_t - \nu(\mathbb{R}^d)t\mathbb{E}[\xi_1].$$

We finish this section with some brief words about the richness of the Lévy process that is encoded in the Lévy measure structure. The Lévy measure ν characterizes the frequency of the jumps and the height distribution of the process. If this measure is infinite, then the process has an infinite number of jumps of arbitrary small sizes in any small interval. The measure ν has no mass at the origin but infinitely many jumps can occur around the origin. Furthermore, the mass away from the origin is bounded. This means that only a finite number of big jumps can occur. The large/moderate deviations principles of **Chapter 2**, **Chapter 3** and **Chapter 4** and consequent use of LDP/MDP asymptotic estimates to study the first exit time problem in **Chapter 2** and **Chapter 3** were stated under the perturbation of the corresponding ODEs by Lévy processes with Lévy measurex absolutely continuous to the Lebesgue measure with an exponentially light density. In particular, the intensity of the process is finite ($\nu(\mathbb{R}^d) < \infty$). This has consequences for the properties of the jumps of the process. The following result states and generalizes mathematically what was told in this paragraph. For a proof we refer the reader to *Sato (1999)*-**Theorem 21.3**.

Proposition B.1.2. *Let $(L_t)_{t\geq 0}$ be a Lévy process with Lévy triplet (b, σ, ν).*

1. *If $\nu(\mathbb{R}^d) < \infty$, then almost all paths of $(L_t)_{t\geq 0}$ have a finite number of jumps on every compact time interval. In that case we say that the Lévy process has finite activity.*

2. *If $\nu(\mathbb{R}^d) = \infty$, then almost all paths of $(L_t)_{t\geq 0}$ have an infinite number of jumps on every compact interval. In that case we say that the Lévy process has infinite activity.*

B.2 Poisson random measures

The stochastic dynamical systems that are considered in this work are perturbations of ODEs by Lévy processes that are expressed in terms of integrals with respect to an underlying random measure. Poisson random measures are the right mathematical idealization to describe the jump properties of a Lévy process.

Example B.2.1 (**A random measure associated to a compund Poisson process**). *Let us define on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a compound Poisson process $(L_t)_{t\geq 0}$ with drift, defined by the identity,*

$$L_t = bt + \sum_{i=1}^{P_t} \xi_i, \quad \textit{for all } t \geq 0.$$

We denote by $(T_i)_{i\in\mathbb{N}}$ the sequence of jump times of the Poisson process $(P_t)_{t\geq 0}$. It follows that $(T_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of exponentially distributed random variables with parameter $\lambda > 0$.
Given $A = (0,t] \times B \in \mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R}^d - \{0\})$ we define

$$M(A) := card\{i \geq 0 \mid (T_i, \xi_i) \in A\} = \sum_{i=1}^{\infty} \mathbf{1}_{A(T_i,\xi_i)}$$

which counts the number of jumps ocurred in the time interval $(0,t]$ located in the Borel set $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$. Since $(L_t)_{t\geq 0}$ has almost surely a finite number of jumps over a finite period of time it follows that $M(A) < \infty$ $\mathbb{P}$-a.s., for all $t \geq 0$ such that $A \in \mathcal{B}([0,t)) \otimes \mathcal{B}(\mathbb{R}^d - \{0\})$.
*It is proved in Kyprianou (2014)-**Lemma 2.2.** that, given $k \geq 1$ and $A_1, \dots, A_k$ disjoint sets of $\mathcal{B}([0,\infty)) \otimes \mathcal{B}(\mathbb{R}^d - \{0\})$, $M(A_1), \dots, M(A_k)$ are independent and Poisson distributed with parameters*

$$\lambda_i = \lambda \int_{A_i} ds\mu(dz), \quad \textit{for all } i \in \{1, \dots, k\}$$

respectively.

We now define now the concept of Poisson random measure that contains the particular case of the measure M constructed above.

Definition B.2.1 (**Poisson random measure**). *Let $(S, \mathcal{S}, \mu)$ be a σ-finite measure space. Let*

$$M : \mathcal{S} \longrightarrow \mathbb{N} \cup \{\infty\},$$

*defined in a way that the family $\{M(A) \mid A \in \mathcal{S}\}$ are random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. M is called a **Poisson random measure** on $\mathcal{S}$ with intensity μ if*

i) for $n \in \mathbb{N}$ and mutually disjoint sets $A_1, \dots, A_n \in \mathcal{S}$, the random variables $M(A_1), \dots, M(A_n)$ are independent;

ii) for every $A \in \mathcal{S}$, $M(A)$ is Poisson distributed with parameter $\mu(A)$ (we allow $\mu(A) \in [0, \infty)$);

iii) M is a measure $\mathbb{P}$-a.s. .

An important notion of convergence we use in this thesis is the **vague convergence**.

Definition B.2.2 (Vague convergence of measures). *Let $\mathcal{X}$ be a locally compact Hausdorff space and $\mathcal{M}(\mathcal{X})$ the space of the locally finite measures defined on the Borel sets of $\mathcal{X}$. We say that a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\mathcal{X})$ converges in the vague topology to $\mu \in \mathcal{M}(\mathcal{X})$ as $n \to \infty$ if*

$$\int_{\mathcal{X}} f(x)\mu_n(dx) \to \int_{\mathcal{X}} f(x)\mu(dx), \qquad \text{for all } f \in C_0(\mathcal{X}),$$

where $C_0(\mathcal{X})$ is the space of the continuous functions with values in $\mathbb{R}$ that vanish in infinity equipped with the uniform norm.

In view of **Skorokhod's representation theorem** it is important to have a non-atomic probability measure. From *Kingsman (1993)*-**Chapter 2-Section 5** we can deduce the following result.

Proposition B.2.1 (Existence of a Poisson random measure). *Let $\mathcal{S}$ be a locally compact Polish space equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{S})$ and fix μ a non-atomic measure. We denote $\mathcal{M}(S)$ the space of the locally finite measures defined on S endowed with the vague topology. There exists a unique non-atomic probability measure $\mathbb{P}$ on $(\mathcal{M}(S), \mathcal{B}(\mathcal{M}(\mathcal{S})))$ such that the canonical map*

$$M : \mathcal{M}(S) \longrightarrow \mathcal{M}(S)$$

is a Poisson random measure with intensity μ.

We now fix a locally compact Polish space S and a σ-finite measure ν defined on the Borel sets of S. We denote by μ the product measure defined on $((0, \infty) \times S, \mathcal{B}((0, \infty) \times S))$ by

$$\mu((0, t] \times A) = t\nu(A), \quad t \geq 0, A \in \mathcal{E}.$$

Following **Proposition B.2.1** let M be a Poisson random measure with intensity μ defined on the probability space $(\mathcal{M}(\mathbb{R}^+ \times S), \mathcal{B}(\mathcal{M}(\mathbb{R}^+ \times S)), \mathbb{P})$. We set $\tilde{M} = M - \mu$. We call $\tilde{M}$ a compensated Poisson random measure with intensity μ. Following *Applebaum (2009)*-**Section 2.4** and *Sato (2013)*-**Chapter 4** we state the **Lévy-Itô decomposition theorem** which characterizes the paths of a Lévy process in the following way.

Theorem B.2.1 (Lévy-Itô decomposition theorem). *Consider $b \in \mathbb{R}^d$, σ a positive-definite matrix of $\mathbb{R}^{d\times d}$ and ν a measure defined on the Borel sets of $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(dz) < \infty$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes exist, L^1, L^2, L^3 and L^4 with the following properties. $L^1(t) =$*

bt, for all $t \geq 0$ is called a constant drift, L^2 is a Brownian motion with covariance $\sqrt{\sigma}$, L^3 is a compound Poisson process, and L^4 is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on every finite time interval. Hence, for $L = L^1 + L^2 + L^3 + L^4$ there exists a probability space on which $(L_t)_{t \geq 0}$ is a Lévy process with characteristic exponent

$$\psi(u) = i\langle b, u\rangle - \frac{1}{2}(\sigma u, u) + \int_{\mathbb{R}^d} (e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle 1_{\{|z|<1\}})\nu(dz), \quad u \in \mathbb{R}^d.$$

Conversely, given a Lévy process defined on a probability space, there exists $b \in \mathbb{R}^d$, a Wiener process $(B)_{t \geq 0}$, with covariance matrix $\sqrt{\sigma} \in \mathbb{R}^{d \times d}$ and an independent Poisson random measure defined on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ such that, for all $t \geq 0$,

$$L_t = bt + \sqrt{\sigma} B_t + \int_0^t \int_{0<|z|<1} z\tilde{M}(dsdz) + \int_0^t \int_{\{|z|>1\}} zM(ds, dz). \tag{B.2.1}$$

Remark B.2.1. *In the notation of the last theorem, for all $t \geq 0$,*

$$\begin{aligned} L_t^1 &= bt, \\ L_t^2 &= \sqrt{\sigma} B_t, \\ L_t^3 &= \int_0^t \int_{|z|>1} zM(ds, dz), \\ L_t^4 &= \int_0^t \int_{0<|z|<1} z\tilde{M}(dsdz). \end{aligned}$$

B.3 Stochastic calculus

Fix $T > 0$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. We assume that the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual hyphothesis of completeness, i.e. F_0 contains all sets of $\mathbb{P}$-measure zero, and right continuity, i.e. $\mathcal{F}_t = \mathcal{F}_{t^+}$, for all $t \in [0,T]$. We denote by $\mathcal{P}$ the σ-field on $\Omega \times [0,T]$ generated by the all left continuous and adapted processes. $\mathcal{P}$ is called the predictable σ-field. A stochastic process $X : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ is called $\mathbb{F}$-predictable if it is $\mathbb{F}$-adapted and $\mathcal{P}$-measurable. If $X : \Omega\times[0,T]\times\mathbb{R}^k \longrightarrow \mathbb{R}^d$, the stochastic process $(X_t)_{t\in[0,T]}$ is called $\mathbb{F}$-predictable if it is $\mathbb{F}$-adapted and $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^k)$-measurable.
We proceed by presenting *Itô's formula.* The proof follows, after localization arguments, as the proof of **Theorem 4.4.13** in *Applebaum (2009)*

Proposition B.3.1 (**Itô's formula**). *Consider a stochastic process $(X_t)_{0\leq t\leq T}$ satisfying the following equation*

$$X_t = X_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t \int_{|z|<1} H(s,z)\tilde{M}(ds,dz) + \int_0^t \int_{|z|\geq 1} K(s,z)M(ds,dz) \quad 0 \leq t \leq T.$$

where $b : [0,T]\times\mathbb{R}^d$, $\sigma : [0,T] \longrightarrow \mathbb{R}^{d\times d}$ and $H, K : [0,T]\times\mathbb{R}^k \longrightarrow \mathbb{R}^d$ are measurable functions such that K is predictable,

$$\int_0^T |b(s)|^2 ds, \int_0^T |\sigma(s)|^2 ds, \int_0^T \int_{|z|<1} |H(s,z)|^2 \nu(dz)ds < \infty.$$

Write the continuous part of $(X_t)_{0\leq t\leq T}$ as

$$X_t^c = X_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s.$$

Let $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$. Then, for any stopping time τ with values in $[0,T]$ we have

$$\begin{aligned}
\varphi(\tau, X_\tau) = \varphi(0, X_0) &+ \int_0^\tau \frac{d}{dt}\varphi(s, X_s)ds + \int_0^\tau \nabla_x\varphi(s, X_{s-})dX_s^c \\
&+ \frac{1}{2}\int_0^\tau \nabla_x\varphi(s, X_{s-})\sigma^2(s)ds \\
&+ \int_0^\tau \int_{|z|\geq 1} (\varphi(s, X_{s-} + K(s,z)) - \varphi(s, X_{s-}))M(ds,dz) \\
&+ \int_0^\tau \int_{|z|<1} (\varphi(s, X_{s-} + H(s,z)) - \varphi(s, X_{s-}))\tilde{M}(ds,dz) \\
&+ \int_0^\tau \int_{|z|<1} (\varphi(s, X_{s-} + H(s,z)) - \varphi(s, X_{s-}) - \nabla_x\varphi(s, X_{s-})H(s,z))\nu(dz)ds
\end{aligned}$$

We use often *Itô's formula* for the product along the text, which we present in the following proposition. The proof follows, after localization arguments, as the proof of **Theorem 4.4.13** in *Applebaum (2009)*

Proposition B.3.2 (Itô's formula for the product). *Consider two stochastic processes $(X_t^i)_{0\leq t\leq T}$, $i\in\{1,2\}$, satisfying the following dynamics*

$$X_t^i = X_0^i + \int_0^t b^i(s)ds + \int_0^t \sigma^i(s)dB_s + \int_0^t\int_{|z|<1} H^i(s,z)\tilde{M}(ds,dz) + \int_0^t\int_{|z|\geq 1} K^i(s,z)M(ds,dz) \quad 0\leq t\leq T.$$

where $b^i : [0,T]\times\mathbb{R}^d$, $\sigma^i : [0,T] \longrightarrow \mathbb{R}^{d\times d}$ and $H^i, K^i : [0,T]\times\mathbb{R}^k \longrightarrow \mathbb{R}^d$ measurable functions such that K^i is predictable and

$$\int_0^T |b^i(s)|^2 ds, \int_0^T |\sigma^i(s)|^2 ds, \int_0^T\int_{|z|<1} |H^i(s,z)|^2\nu(dz)ds < \infty < \infty.$$

Let $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$. Then, for any stopping time τ with values in $[0,T]$ we have

$$\langle X_\tau^1, X_\tau^2\rangle = \langle X_0^1, X_0^2\rangle + \int_0^\tau \langle X_{s-}^1, dX_s^2\rangle + \int_0^\tau \langle X_{s-}^2, dX_s^1\rangle + [X_\tau^1, X_\tau^2],$$

where the quadratic covariation is given by,

$$[X_t^1, X_t^2] = \int_0^t \langle\sigma^1(s),\sigma^2(s)\rangle ds + \int_0^t\int_{|z|<1} \langle H^1(s,z), H^2(s,z)\rangle M(ds,dz) + \int_0^t\int_{|z|\geq 1} \langle K^1(s,z), K^2(s,z)\rangle M(ds,dz) \quad t\geq 0.$$

Proposition B.3.3 (Stochastic inequalities).

1. ***Chebyseff-Markov's inequality***. *Given a random variable $X:\Omega\longrightarrow\mathbb{R}^+$ and any $\alpha>0$ we have*

   $$\mathbb{P}(X\geq\alpha) \leq \frac{\mathbb{E}[X]}{\alpha}.$$

   *For a proof we refer the reader to Klenke (2014)-**Theorem 5.11**.*

2. ***Burkholder-Davis-Gundy's inequalities.*** *Let $(M_t)_{t\in[0,T]}$ be a local martingale and $([M]_t)_{t\geq 0}$ be the quadratic variation process. For any $p\geq 1$ there exist constants $k,K>0$ depending on p but independent of $(M_t)_{t\in[0,T]}$ such that*

   $$\mathbb{E}\Big[\sup_{0\leq t\leq T}|M_t|^p\Big] \leq k\mathbb{E}\Big[[M]_T^{\frac{p}{2}}\Big] \leq K\mathbb{E}\Big[\sup_{0\leq t\leq T}|M_t|^p\Big].$$

   *For a proof we refer the reader to Protter (2004)-**Theorem IV.48**.*

From *Sato (1999)*-**Theorem 25.3** we have the following characterization of the moments of a Lévy process.

Proposition B.3.4. *Given a Lévy process $(L_t)_{t\geq 0}$ with Lévy triplet (b, σ, ν), $(L_t)_{t\geq 0}$ has pth-moment ($\mathbb{E}[|L_t|^p] < \infty$) for all $t \geq 0$ and for some $p \geq 0$ if and only if*

$$\int_{|z|\geq 1} |z|^p \nu(dz) < \infty.$$

From *the Lévy-Itô decomposition theorem* (**Theorem B.2.1**), given a Lévy process with triplet (b, σ, ν), with values in $\mathbb{R}^d$, we have the following representation

$$L_t = bt + \sqrt{\sigma} B_t + \int_0^t \int_{|z|<1} z\tilde{M}(dsdz) + \int_0^t \int_{|z|\geq 1} zM(dsdz), \quad t \geq 0,$$

Here $(B_t)_{t\geq 0}$ is a Brownian motion and M a independent Poisson random measure on $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d - \{0\})$. If we assume that $(L_t)_{t\geq 0}$ has first moment, from the last proposition, the representation above turns into

$$L_t = at + \sqrt{\sigma} B_t + \int_0^t \int_{\mathbb{R}^d - \{0\}} z\tilde{M}(dsdz),$$

where $a = b + \displaystyle\int_{|z|\geq 1} z\nu(dz)$. It is natural to ask if such a representation holds for more generic stochastic process in terms of a fixed Brownian motion and a independent Poisson random measure defined on the same probability space. This is the content of the *martingale representation theorem* stated in the next theorem. For a proof we refer the reader to *He et al. (1992)*-**Theorem 11.31** and *He et al. (1992)*-**Corollary 11.32**.

Theorem B.3.1 (The martingale representation theorem). *Fix $T > 0$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which it is defined a Brownian motion $(B_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ and an independent compensated Poisson random measure $\tilde{M}$ over the Borel σ-field $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d - \{0\})$. Let $\mathbb{F}$ be the natural filtration generated by the two processes. Let $(M_t)_{t\geq 0}$ be a $\mathbb{F}$-local square integrable martingale. Then there exist two $\mathbb{F}$-predictable processes $(Z_s)_{s\in[0,T]}$ and $(V_s(z))_{\{s\in[0,T], z\in\mathbb{R}^d-\{0\}\}}$ integrable with respect to $(B_t)_{t\in[0,T]}$ and M respectively, satisfying*

$$\mathbb{E}\Big[\int_0^T |Z_s|^2 ds\Big] < \infty \quad \textit{and} \quad \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^d} |V_s(z)|^2 \nu(ds) ds\Big] < \infty$$

and such that

$$M_t = M_0 + \int_0^t Z_s dB_s + \int_0^t \int_{\mathbb{R}^d - \{0\}} V_s(z)\tilde{M}(dsdz), \quad t \in [0, T], \quad \mathbb{P}-\ a.s.$$

We state a specific form of *Girsanov's theorem* that characterizes the change of measure on a probability space under which a Lévy process remains a process with independent increments under the new measure. The result follows from *Jacod and Shiryaev (1987)*-**Theorem III.3.24**.

Theorem B.3.2 (**Girsanov Theorem**). *Fix $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by two independent processes, a Brownian motion, $(B_t)_{t\in[0,T]}$ with values in $\mathbb{R}^d$, and a Poisson random measure M defined on $\mathcal{B}([0,T] \times \mathbb{R}^d - \{0\})$, with compensator given by $ds \otimes \nu$.*
Let $f : [0,T] \longrightarrow \mathbb{R}^d$ and $\alpha : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be measurable functions such that

$$\int_0^T \int_{\mathbb{R}^d} |ze^{\alpha(s,z)} - 1|\nu(dz)ds < \infty \quad \text{and} \qquad\qquad \int_0^T |f(s)|^2 ds < \infty.$$

For every $t \in [0,T]$ define

$$\begin{aligned} Z_t^1 &:= \exp\Big(- \int_0^t f(s)dB_s - \frac{1}{2}\int_0^t |f(s)|^2 ds\Big); \\ Z_t^2 &:= \exp\Big(- \int_0^t \int_{\mathbb{R}^d} e^{\alpha(s,z)-1}\nu(dz)ds + \int_0^t \int_{\mathbb{R}^d} \alpha(s,z)M(ds,dz);\Big) \text{ and} \\ Z_t &:= Z_t^1 Z_t^2. \end{aligned}$$

Then $(Z_t)_{t\in[0,T]}$ is an $\mathbb{F}$-martingale and $\mathbb{E}[Z_t] = 1$ for all $t \in [0,T]$. Define a new probability measure given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T.$$

Under $\mathbb{Q}$ the process $\Big(B_t + \int_0^t f(s)ds\Big)_{t\in[0,T]}$ is a Brownian motion and M is a Poisson random measure with compensator $\hat{\nu}(ds,dz) = e^{\alpha(s,z)}\nu(dz)ds$.

B.4 Proof of Theorem 1.1.1

*Proof of **Theorem 1.1.1**.* We fix the probability space $(\Omega, \mathcal{F}, \bar{\mathbb{P}})$ defined in the first section of **Chapter 1** with the completion $(\bar{\mathcal{F}}_t)_{t\geq 0}$ of the filtration generated by $\bar{N}$. For every $\varepsilon > 0$, we consider the SDE

$$X_t^\varepsilon = x - \int_0^t \nabla U(X_s^\varepsilon)ds + \varepsilon \int_0^t \int_{\mathbb{R}^d} z\tilde{N}^{\frac{1}{\varepsilon}}(ds, dz), \quad t \geq 0. \tag{B.4.1}$$

under **Condition 1.1.1**. Given $\varepsilon > 0$, the measure $\tilde{N}^{\frac{1}{\varepsilon}}$ is the compensated Poisson random measure defined on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ over $\mathcal{B}([0,\infty) \times \mathbb{R}^d)$ with compensator given by $\dfrac{1}{\varepsilon} ds \otimes \nu$. The measure ν is of the form

$$\nu(dz) = e^{-|z|^\alpha} dz, \quad \text{for some } \alpha > 0,$$

1. We assume in a first step that ∇U is global Lipschitz, i.e. there exists $C > 0$ such that for every $x, y \in \mathbb{R}^d$

$$|\nabla U(x) - \nabla U(y)| \leq C|x - y|,$$

We consider the following functional space

$$\mathcal{V} := \Big\{ x : \Omega \times [0,T] \longrightarrow \mathbb{R}^d \mid (x_t)_{t\in[t\geq 0]} \text{ is a stochastic process} \\ \text{satisfying (i), (ii) and (iii)} \Big\},$$

where the conditions (i), (ii) and (iii) are the following:

i) the map $\omega \mapsto x_t(\omega)$ is $\bar{\mathcal{F}}_t$-measurable, for every $t \geq 0$;

ii) the process x is stochastically continuous;

iii) for a positive $\rho > 0$ that will be fixed later

$$||x||_{\mathcal{V}} = \left(\bar{\mathbb{E}}\Big[\sup_{t\geq 0} e^{-2\rho t} |x_t|^2 \Big] \right)^{1/2} < \infty.$$

$(\mathcal{V}, ||.||_{\mathcal{V}})$ is a Banach space. Condition (i) could be replaced by requiring x to be predictable, since there are no deterministic jumps due to the condition of stochastic continuity (ii). Condition (ii) could be replaced by requiring that the process x is a càdlàg process $\bar{\mathbb{P}}$- a.s. . It is immediate that $\mathcal{V}$ is identified with the space of càdlàg adapted processes.

For every $\varepsilon > 0$, we consider the nonlinear mapping

$$(T^\varepsilon y)(t) = x - \int_0^t \nabla U(y_s)ds + \varepsilon \int_0^t \int_{\mathbb{R}^d} z\tilde{N}^{\frac{1}{\varepsilon}}(ds, dz), \quad t \geq 0.$$

Since $\nabla U(0) = 0$ we have

$$\begin{aligned}
\Big|\Big| \int_0^{\cdot} \nabla U(y_s) ds \Big|\Big|_{\mathcal{V}}^2 &= \bar{\mathbb{E}}\Big[\sup_{t\geq 0} e^{-2\rho t} \Big| \int_0^t \nabla U(y_s) ds \Big|^2 \Big] \\
&\leq \bar{\mathbb{E}}\Big[\sup_{t\geq 0} \int_0^t e^{-2\rho(t-s)} e^{-2\rho s} |\nabla U(y_s)|^2 ds \Big] \\
&\leq \frac{1}{2\rho} \bar{\mathbb{E}}\Big[\sup_{s\geq 0} e^{-2\rho s} |\nabla U(y_s)|^2 \Big] \\
&\leq \frac{C^2}{2\rho} \mathbb{E}\Big[\sup_{s\geq 0} e^{-2\rho s} |y_s|^2 \Big] \\
&< \infty.
\end{aligned}$$

Furthermore, it follows

$$\begin{aligned}
\varepsilon^2 \bar{\mathbb{E}}\Big[\sup_{t\geq 0} e^{-2\rho t} \Big| \int_0^t \int_{\mathbb{R}^d} z \tilde{N}^{\frac{1}{\varepsilon}}(ds, dz) \Big|^2 \Big] &\leq \varepsilon^2 \bar{\mathbb{E}}\Big[\Big| \sup_{t\geq 0} \int_0^t \int_{\mathbb{R}^d} e^{-\rho s} z \tilde{N}^{\frac{1}{\varepsilon}}(ds, dz) \Big|^2 \Big] \\
&= \varepsilon \sup_{t\geq 0} \int_0^t \int_{\mathbb{R}^d} e^{-2\rho s} |z|^2 \nu(dz) dz \\
&\leq \frac{\varepsilon c_\nu^2}{2\rho} \\
&< \infty,
\end{aligned}$$

where $c_\nu^2 := \int_{\mathbb{R}^d} |z|^2 \nu(dz) < \infty$.

Hence, we conclude that $T^\varepsilon \mathcal{V} \longrightarrow \mathcal{V}$.
For every $\varepsilon > 0$ and $x, y \in \mathcal{V}$, using the Lipschitz conditions on ∇U, we derive

$$||T^\varepsilon x - T^\varepsilon y||_{\mathcal{V}}^2 \leq \frac{C^2}{2\rho} ||x - y||_{\mathcal{V}}^2.$$

Choosing $\rho > \frac{C^2}{2}$ implies that T^ε is a contraction on $\mathcal{V}$ and *Banach's fixed point theorem* yields the existence and uniqueness of a fixed point X^ε for T^ε which is a solution of (B.4.1). The integral equation satisfied by X^ε implies that X^ε is an adapted process with càdlàg paths.

2. We now show the uniqueness of solution of (B.4.1). For every $\varepsilon > 0$ let $(X_t^\varepsilon)_{t\geq 0}$ and $(Y_t^\varepsilon)_{t\geq 0}$ be two solutions of (B.4.1). For every $n \in \mathbb{N}$ we define the stopping times

$$\begin{aligned}
\tau_n &:= \inf\{t \geq 0 \mid |X_t^\varepsilon| \geq n\} \\
\tau_n' &:= \inf\{t \geq 0 \mid |Y_t^\varepsilon| \geq n\} \text{ and} \\
\tau_n^* &:= \tau_n \wedge \tau_n'.
\end{aligned}$$

We have that $\tau_n^* \to \infty$ as $n \to \infty$ $\bar{\mathbb{P}}$-a.s. For every $T > 0$ we have

$$X^\varepsilon_{T\wedge\tau_n^*} - Y^\varepsilon_{T\wedge\tau_n^*} = \int_0^{T\wedge\tau_n^*} (-\nabla U(X^\varepsilon_s) + \nabla U(Y^\varepsilon_s))ds$$

and using *Gronwall's inequality* we conclude for every $n \in \mathbb{N}$ and $T > 0$

$$\bar{\mathbb{P}}\Big(X^\varepsilon_{T\wedge\tau_n^*} = Y^\varepsilon_{T\wedge\tau_n^*}\Big) = 1.$$

Letting $n, T \to \infty$ the result follows.

3. We consider now the case ∇U is a locally Lipschitz function. For every $N \in \mathbb{N}$ we define the function ψ_N in the following way:

$$\psi_N(z) := \begin{cases} z & \text{if } |z| \le N; \\ N\frac{z}{|z|} & \text{if } |z| > N. \end{cases}$$

For every $N \in \mathbb{N}$ we define the function $(\nabla U)_N(z) = \nabla U(\psi_N(z))$ for all $z \in \mathbb{R}^d$ and we consider the corresponding solution $(X^{\varepsilon,N}_t)_{t\ge0}$ of (B.4.1) when ∇U is replaced by $(\nabla U)_N$. Furthermore, for every $N \in \mathbb{N}$ we define the stopping time

$$\tau_N := \inf\{t \ge 0 \mid |X^{\varepsilon,N}_t| \ge N\}.$$

By the uniqueness property proved before we conclude that $X^{\varepsilon,N}_t = X^{\varepsilon,N+1}_t$ for every $t \in [0, \tau_N]$.

For every $\varepsilon > 0$ and $t \le \tau_N$ we define $X^{\varepsilon,\infty}_t := X^{\varepsilon,N}_t$. It follows that $X^{\varepsilon,\infty}$ solves pathwise (B.4.1) for all $t \ge 0$ if we prove that $\tau_N \to \infty$ as $n \to \infty$ $\bar{\mathbb{P}}$-a.s. With analogous arguments used in the proof of **Proposition 2.2.1** we conclude that for every $\varepsilon > 0$

$$\bar{\mathbb{E}}\Big[\sup_{N\in\mathbb{N}}\sup_{t\ge0} e^{-2\rho t}|X^{\varepsilon,N}_t|^2\Big] < \infty.$$

Chebyshev's inequality implies that $\bar{\mathbb{P}}(\sup_{t\ge0}|X^{\varepsilon,N}_t| > N) \to 0$ as $N \to \infty$. This finishes the proof.

□

Appendix C

Weak convergence of probability measures and the space of càdlàg functions

We present here some classical notions and results about weak convergence of probability measures and about the space $\mathbb{D}([0,T],\mathbb{R}^d)$ that are used along our work in a direct or indirect way. Weak convergence methods play a crucial role in the derivation of the large/moderate deviations principles obtained in **Chapter 2**, **Chapter 3** and **Chapter 4**, specially in the verification of tightness properties for the controlled processes. The topological structure of the càdlàg space $\mathbb{D}([0,T],\mathbb{R}^d)$ is used several times in **Chapter 2**, **Chapter 3** and **Chapter 4**, specially in the asymptotic study of the first exit time associated to the jump-diffusion.
All definitions and results can be found in **Chapter 1** and **Chapter 3** in *Billinsgley (1999)* which is the major reference for this part of the appendix.

C.1 Convergence in distribution, weak convergence and tightness

Let $(\mathcal{S}, d)$ be a metric space.

Definition C.1.1 (**Weak convergence of probability measures**). *Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be a sequence of probability measures defined on the Borel sets of $(\mathcal{S}, d)$ and $\mathbb{P}$ some probability measure also defined on the Borel sets of $(\mathcal{S}, d)$. We say that $(\mathbb{P}_n)_{n\in\mathbb{N}}$ converges weakly to $\mathbb{P}$ as $n \to \infty$ if, for any bounded continuous function $f \in C_b(\mathcal{S})$,*

$$\int_{\mathcal{S}} f(x)\mathbb{P}_n(dx) \to \int_{\mathcal{S}} f(s)\mathbb{P}(dx), \quad \text{as } n \to \infty.$$

We write $\mathbb{P}_n \Rightarrow \mathbb{P}$ as $n \to \infty$.

Definition C.1.2 (**Weak convergence of random variables**). *For any $n \in \mathbb{N}$, let X and X_n, for any $n \in \mathbb{N}$, be $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$- valued random variables defined on the probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ respectively. We say that $(X_n)_{n\in\mathbb{N}}$ converges weakly or in law to X if, for any bounded continuous function $f \in C_b(\mathcal{S})$,*

$$\mathbb{E}_n(f(X_n)) \to \mathbb{E}(f(X)), \quad \text{as } n \to \infty,$$

where $\mathbb{E}$ and $\mathbb{E}_n$ are respectively the expectation operators defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. We write $X_n \Rightarrow X$ as $n \to \infty$.

For every μ Borel measure defined on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ we can associate the linear functional

$$\varphi_\mu(f) = \int_{\mathcal{S}} f(x)\mu(dx), \quad \text{for all } f \in C_b(\mathcal{S}).$$

Conversely if $(\mathcal{S}, d)$ is compact every positive bounded linear functional on $C(\mathcal{S}) = C_b(\mathcal{S})$ is represented by a finite Borel measure on $\mathcal{S}$. This the content of the *Riesz representation theorem*.

Proposition C.1.1 (**Riesz representation theorem**). *Let $(\mathcal{S}, d)$ be a compact Hausdorff space and φ a positive bounded linear functional defined on $C_b(\mathcal{S})$ and such that $||\varphi||_\infty = 1$. Then there exists a unique probability measure $\mathbb{P}$ on $\mathcal{S}$ such that*

$$\langle \varphi, f \rangle_{C_b(\mathcal{S})} = \int_{\mathcal{S}} f(x)\mathbb{P}(dx), \quad \text{for all } f \in C_b(\mathcal{S}).$$

The $\langle ., . \rangle_{\mathcal{S}}$ denotes the dual pairing of $C_b(\mathcal{S})$. A proof can be found in *Rudin (1987)* - **Theorem 2.14**. Using a scaling argument *Riesz representation theorem* can be extended to a correspondence between not necessarily normalized positive bounded functionals on $C(\mathcal{S}) = C_b(\mathcal{S})$ and finite measures defined on the Borel sets of $\mathcal{S}$. The representation theorem by a measure can be extended to every member of $C_b(\mathcal{S})$, but it uses signed measures.

In *Rudin (1987)* this is studied with detail.

The concept of weak convergence of probability measures and weak* convergence are related in an intrinsic way when $(\mathcal{S}, d)$ is a compact metric space. From *Riesz representation theorem* the following observation is immediate.

Proposition C.1.2. *Let $(\mathcal{S}, d)$ be a compact metric space, $\mathbb{P}$ and $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be probability measures on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. Then the following two statements are equivalent, as $n \to \infty$,*

1. $\mathbb{P}_n \Rightarrow \mathbb{P}$;
2. $\varphi_{\mathbb{P}_n} \rightharpoonup \varphi_{\mathbb{P}}$.

The following notion of convergence for probability measures implies weak convergence.

Definition C.1.3. *If for every $n \in \mathbb{N}$, $\mathbb{P}_n$ and $\mathbb{P}$ are probability measures on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, $\mathbb{P}_n$ converges in total variation to $\mathbb{P}$ and we write $\mathbb{P}_n \to^{TV} \mathbb{P}$ as $n \to \infty$ if*

$$\sup_{A\in\mathcal{B}(S)} |\mathbb{P}_n(A) - \mathbb{P}(A)| \to \quad \text{as } n \to \infty.$$

Proposition C.1.3 (Scheffe's theorem). *For every $n \in \mathbb{N}$, let $\mathbb{P}_n$ and $\mathbb{P}$ be probability measures on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ with densities f_n and f respectively with respect to a certain measure μ defined on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. If $f_n \to f$ μ-a.e. as $n \to \infty$, then $\mathbb{P}_n \to^{TV} \mathbb{P}$ and therefore $\mathbb{P}_n \Rightarrow \mathbb{P}$.*

Proof. For any $A \in \mathcal{B}(\mathcal{S})$ it follows

$$\begin{aligned}
|\mathbb{P}_n(A) - \mathbb{P}(A)| &= \Big| \int_A (f_n(x) - f(x))\mu(dx)\Big| \\
&\leq \int_{\mathcal{S}} |f_n(x) - f(x)|\mu(dx) \\
&= 2\int_{\mathcal{S}} (f(x) - f_n(x))^+\mu(dx),
\end{aligned}$$

We use dominated convergence to conclude that

$$\int_{\mathcal{S}} (f(x) - f_n(x))^+\mu(dx) \to 0 \text{ as } n \to \infty,$$

and the result follows. □

Definition C.1.4 (Tightness). *A family Π of probability measures on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ is called tight if for every $\delta > 0$ there exists a compact set $K^\delta \subset \mathcal{S}$, such that*

$$\mathbb{P}(K) < 1 - \delta, \qquad \textit{for all } \mathbb{P} \in \Pi.$$

Proposition C.1.4. *If $\mathcal{S}$ is a separable complete metric space every $\mathbb{P}$ probability measure defined on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ is tight.*

Definition C.1.5 (Relatively compactness). *A family of probability measures* Π *defined on* $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ *is said to be relatively compact if any sequence in* Π *contains a weakly convergent subsequence. Although,* Π *may not be closed.*

The two definitions above are equivalent. This is the content of the next theorem. For a proof we refer **section 5** in *Billingsley (1999)*.

Proposition C.1.5 (Prokhorov's theorem). *A family of probability measures* Π *on* $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ *is tight if and only if it is relatively compact.*

We present the classic version of Skorokhod's representation theorem that can be found in **section 6** in *Billingsley (1999)*.

Proposition C.1.6 (Skorokhod's representation theorem). *Let* $\mathbb{P}$ *and* $\mathbb{P}_n$*,* $n \in \mathbb{N}$*, be probability measures on* $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$*. Suppose that* $P_n \Rightarrow P$ *as* $n \to \infty$ *and that* $\mathbb{P}$ *has a separable support. Then, for all* $n \in \mathbb{N}$*, there exist random variables* X_n *and* X *defined on a common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $\mathbb{P}_n$ *is the law of* X_n*,* $\mathbb{P}$ *is the probability distribution of* X *and* $X_n \to X$*,* $\mathbb{P}$*-a.s. as* $n \to \infty$

In the setting of our work we will have always the situation that $(X_n)_{n\in\mathbb{N}}$ is a family of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathcal{S}$ such that $\mathbb{P} \circ (X_n)^{-1} \Rightarrow \mu$ as $n \to \infty$. We would like to guarantee the existence of an $\mathcal{S}$-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that X has law μ. The following result answers this question positively if we require $\mathbb{P}$ to be non-atomic and μ separable. For a proof we refer *Berti et al. (2007)* -**Theorem 3.1**.

Proposition C.1.7. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a non-atomic probability space and* μ *a separable probability measure on* $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$*. Then if* μ *is tight, we conclude that* μ *is the law of some* S*-valued random variable* X *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

We finish this section with a criteria of convergence for random series, known in the literature as Kolmogorov's 3 series theorem. This result is used in **Chapter 2** and **Chapter 3** to derive a crucial asymptotic estimate for the first exit time of the jump-diffusion from a ball .

Proposition C.1.8 (Variance criteria for series (Khinchin and Kolmogorov)). *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of independent random variables defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with mean zero and* $\sum_{n\in\mathbb{N}} \mathbb{E}[|X_n|^2] < \infty$*. Then,*

$$\sum_{n\in\mathbb{N}} X_n < \infty, \quad \mathbb{P} - a.s.$$

For a proof we refer to *Kallenberg (2002)*- **Lemma 3.1.6**.

Proposition C.1.9 (Kolmogorov's 3 series theorem). *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of independent random variables defined on a common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with expected values* $\mathbb{E}[X_n] = a_n$ *and variances* $var[X_n] = \sigma_n^2$ *such that*

$$\sum_{n\geq 0} a_n < \infty \quad \textit{and} \quad \sum_{n\geq 0} \sigma_n^2 < \infty.$$

Then $\sum_{n\geq 0} X_n < \infty$ $\mathbb{P}$-*a.s.*

Proof. The statement follows from the previous proposition. We observe that $var(X_n - a_n) = 0$ and therefore we can assume $a_n = 0$, for all $n \in \mathbb{N}$. $\square$

C.2 The space of càdlàg functions

The results from this section can be found inBillingsley (1999)-**Chapter 3** and in *Jakubowski (2007).*

Proposition C.2.1. *Every $x \in \mathbb{D}([0,T],\mathbb{R}^d)$ has at most a countable number of jumps. Moreover, x is bounded. Hence, x can be uniformly approximated by linear combinations of indicator functions of intervals which turns x a Borel measurable function.*

The following example shows that the uniform metric is not appropriate for the space $\mathbb{D}([0,T],\mathbb{R}^d)$.

Example C.2.1. *Consider, for all $t \in [0,1]$ $x(t) = \mathbf{1}_{[a,1]}(t)$ and $y(t) = \mathbf{1}_{[b,1]}(t)$ for some $a, b \in [0,1]$. If $a \neq b$, then $||x-y||_\infty = 1$ even if a and b are close, which implies that the uniform metric turns $\mathbb{D}([0,1],\mathbb{R}^d)$ in a non-separable space.*

We introduce the following two metrics on $\mathbb{D}([0,T],\mathbb{R}^d)$.

Definition C.2.1 (**Two metrics in** $\mathbb{D}([0,T],\mathbb{R}^d)$)**.** *We define*

$$\Lambda : \Big\{\lambda : [0,T] \longrightarrow [0,T] \mid \lambda \textit{ increasing homeomorphism}\Big\}.$$

and for all $x, y \in \mathbb{D}([0,T],\mathbb{R}^d)$,

$$d_{J_1}(x,y) = \inf_{\lambda\in\Lambda}\Big(\sup_{t\in[0,T]}|\lambda(t)-t| + \sup_{t\in[0,T]}|x(\lambda(t))-y(t)|\Big),$$
$$d_0(x,y) = \inf_{\lambda\in\Lambda}\Big(\sup_{t\in[0,T]}\Big|\ln\frac{\lambda(t)-\lambda(s)}{t-s}\Big| + \sup_{t\in[0,T]}|x(\lambda(t))-y(t)|\Big).$$

It can be show that d_{J_1} and d_0 are metrics in $\mathbb{D}([0,T],\mathbb{R}^d)$, both turning $\mathbb{D}([0,T],\mathbb{R}^d)$ into a separable space. Furthermore, $\mathbb{D}([0,T],\mathbb{R}^d)$ is a complete metric space under d_0. Hence, $\mathbb{D}([0,T],\mathbb{R}^d)$ is a Polish space. Both metrics are equivalent and they induce the same topology, called *Skorokhod topology.* We refer to *Billingsley (1999)* - **Theorem 12.1** and **Theorem 12.2** for a proof.
We use the metric d_{J_1}. Given x and x_n in $\mathbb{D}([0,T],\mathbb{R}^d)$, for every $n \in \mathbb{N}$, we say that x_n converges to x in the J_1-metric (and in the *Skorokhod topology*) if and only if $d_{J_1}(x_n,x) \to 0$ as $n \to \infty$. Equivalently $x_n \to x$ in the *Skorokhod topology* if there exists a sequence of increasing homeomorphisms $\lambda_n : [0,T] \longrightarrow [0,T]$, such that

$$\sup_{t\in[0,T]}|\lambda_n(t)-t| \to 0, \quad \text{as } n\to\infty \quad \text{and}$$
$$\sup_{t\in[0,T]}|x_n(\lambda_n(t))-x(t)| \to 0, \quad \text{as } n\to\infty.$$

By definition of convergence in the J_1-metric we can immediately conclude that convergence in the uniform norm implies convergence in J_1-metric. The following proposition shows that Skorokhod convergence of elements in $\mathbb{D}([0,T],\mathbb{R}^d)$ implies pointwise convergence for continuity points. Moreover, if the limit is continuous the Skorokhod convergence implies uniform convergence.

Proposition C.2.2. *Let $x, x_n \in \mathbb{D}([0,T],\mathbb{R}^d)$ for every $n \in \mathbb{N}$. If $d_{J_1}(x_n, x) \to 0$ as $n \to \infty$ then x_n converges pointwise to x in the continuity points of x. Furthermore, if $x \in C([0,T],\mathbb{R}^d)$, then $||x_n - x||_\infty \to 0$ as $n \to \infty$.*

Proof. Let $x \in \mathbb{D}([0,T],\mathbb{R}^d)$ and $x_n \in \mathbb{D}([0,T],\mathbb{R}^d)$ for every $n \in \mathbb{N}$. Let us suppose that $d_{J_1}(x_n, x) \to 0$ as $n \to \infty$. Then there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \Lambda$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\lambda_n(t) - t| = \lim_{n\to\infty} |x_n(\lambda_n(t) - x(t))| = 0.$$

By these relations, if t is a point of continuity of x,

$$|x_n(t) - x(t)| \leq |x_n(t) - x(\lambda_n(t))| + |x(\lambda_n(t)) - x(t)| \to 0, \text{as } n \to \infty.$$

Consequently, the result follows. □

In the derivation of the large/moderate deviations we use a useful tigthness criteria in $\mathbb{D}([0,T],\mathbb{R}^d)$ that we state below. This criteria uses the following concept of tightness, so called C-tightness.

Definition C.2.2 (**C-tightness**). *A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{D}([0,T],\mathbb{R}^d)$ is said to be C-tight if it is tight and if all limit points of the sequence $(\mathbb{P}\circ(X_n)^{-1})_{n\in\mathbb{N}}$ are laws of continuous processes, i.e. if a subsequence $(\mathbb{P}\circ(X_{n_k}^{-1}))_{k\in\mathbb{N}}$ converges to a limit probability measure $\mathbb{Q}$ in the space of probability measures over $\mathbb{D}([0,T],\mathbb{R}^d)$, then $\mathbb{Q}$ charges only $C([0,T],\mathbb{R}^d)$.*

For a proof of the following proposition we refer the reader to *Kallianpur and Xiong (1995)*-**Theorem 6.1.1**.

Proposition C.2.3 (**Tightness criteria**). *For every $n \in \mathbb{N}$, let $\mathbb{P}^n$ be a probability measure on $\mathbb{D}([0,T],\mathbb{R}^d)$ induced by a $\mathbb{R}^d$-valued semimartingale $M_n^0 + M_t^n + A_t^n$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, where M_n^0 is a random variable, $(M_t^n)_{t\in[0,T]}$ is a martingale and $(A_t^n)_{t\in[0,T]}$ is a process of finite variation. If the sequence $(M_n^0)_{n\in\mathbb{N}}$ is tight, $([M^n])_{n\in\mathbb{N}}$ and $(A^n)_{n\in\mathbb{N}}$ are C-tight, then $(\mathbb{P}^n)_{n\in\mathbb{N}}$ is tight.*

Appendix D

A primer on large deviations

D.1 Definitions and basic results

From the definition of **large deviations principle** stated in **Definition 1.0.1** it follows the immediate remarks.

Remark D.1.1. *The use of closure and interior of a Borel set $A \in \mathcal{B}(\mathcal{S})$ in the bounds of **Definition 1.0.1** is explicitly necessary if we assume $(X^\varepsilon)_{\varepsilon>0}$ be a family of non-atomic random variables. $(X^\varepsilon)_{\varepsilon>0}$ constitutes a family of non-atomic family of random variables if we have*

$$\mathbb{P}(X^\varepsilon = x) = 0, \qquad \textit{for all } x \in \mathcal{S} \textit{ and for every } \varepsilon > 0.$$

Proof. We prove the statement for the lower bound. Assume that the lower bound in **Definition 1.0.1** holds with int$A$ replaced by A. Then for every $x \in \mathcal{S}$ we have that

$$-I(x) = -\inf_{x \in \{x\}} I(x) \leq \liminf_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon = x) = -\infty,$$

hence $I(x) = \infty$ for all $x \in \mathcal{S}$. Since cl$X = X$, using the upper bound of **Definition 1.0.1** we conclude the absurde

$$0 = \limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon = x) \leq -\inf_{x \in \mathcal{S}} I(x) = -\infty.$$

Similarly we end up with a contradiction if we substitute cl$A$ for A in the upper bound of **Definition 1.0.1** and if we assume the family $(X^\varepsilon)_{\varepsilon>0}$ to be non atomic. □

Remark D.1.2. *The definition of large deviations principle given in **Definition 1.0.1** is equivalent if we replace the upper bound and lower bound of the definition, by respectively the following:*

a) for every closed set $F \subset \mathcal{S}$ we have

$$\limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x \in F} I(x).$$

b) for every open set $G \subset \mathcal{S}$ we have

$$\liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in G) \geq -\inf_{x\in G} I(x)$$

Proof. The large deviations principle upper bound and lower bound imply respectively the upper bound and lower bound of **Remark D.1.2**. Conversely assume (a) and (b) of the remark. Let $A \in \mathcal{B}(\mathcal{S})$. Obviously clA is a closed set and int$A$ is a open set. Using a) we have

$$\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in A) \leq \limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in \mathrm{cl}A) \leq -\inf_{x\in \mathrm{cl}A} I(x).$$

Condition b) yields

$$\liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in A) \geq \liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in \mathrm{int}A) \geq -\inf_{x\in \mathrm{int}A} I(x),$$

which finishes the proof of this remark. □

For sake of completeness of this text we present the definition of regular Hausdorff space.

Definition D.1.1 (**Regular Hausdorff space**). *A topological space $\mathcal{S}$ is a* ***Hausdorff space*** *if for every distinct points $x, y \in \mathcal{S}$ there exist $A, B \subset X$ open sets such that $x \in A$ and $y \in B$.*
A Hausdorff space is regular *if for every closed set $F \subset \mathcal{S}$ and any point $x \in F^c$ we can find disjoint open sets $A, B \subset \mathcal{S}$ such that $F \subset A$ and $x \in B$.*

For the proof of uniqueness of the rate function I in a regular Hausdorff space we use the following property of lower semicontinuous functions in regular Hausdorff spaces and which proof can be found in *Dembo Zeitoni (1998)*-pp. 102-103.

Claim D.1.1. *Let $\mathcal{S}$ be a regular Hausdorff space and $x \in \mathcal{S}$.*

i) For any neighborhood O of x there exists a neighborhood V of x such that cl$V \subset O$.

ii) Any lower semicontinuous function $f : \mathcal{S} \to \bar{\mathbb{R}}$ satisfies

$$f(x) = \sup\{\inf_{y\in A} f(y) \mid A \textit{ is a neighborhood of } x\}.$$

This implies that for every $y \in \mathcal{S}$ and for every $\delta > 0$ we can find $G(y, \delta)$ neighborhood of y, such that

$$(f(y) - \delta) \wedge \frac{1}{\delta} \leq \inf_{z\in G(y,\delta)} f(z).$$

Proposition D.1.1. ***(Uniqueness of the good rate function.)***
Let $\mathcal{S}$ be a Hausdorff regular space. The good rate function I associated to the large deviations principle of $(X^\varepsilon)_{\varepsilon>0}$ with speed $b(\varepsilon)$ of ***Definition 1.0.1*** *is unique.*

Proof. Let us assume the existence of two good rate functions I_1 and I_2 such that the family $(X^\varepsilon)_{\varepsilon>0}$ satisfies the large deviations principle with speed $b(\varepsilon)$. Without loss of generality we assume that there exists $x_0 \in \mathcal{S}$ such that $I_1(x_0) > I_2(x_0)$. We fix $\delta > 0$. From (ii) of **Claim D.1.1** there exists $G(x_0,\delta)$ neighborhood of x_0 such that

$$(I_1(x_0) - \delta) \wedge \frac{1}{\delta} \leq \inf_{z\in G(x_0,\delta)} I_1(z).$$

Due to the statement (i) of **Claim D.1.1** there exists a open set $F(x_0,\delta)$ of x_0 such that $\mathrm{cl}F(x_0,\delta) \subset G(x_0,\delta)$ and we obtain

$$\inf_{z\in \mathrm{cl}F(x_0,\delta)} I_1(z) \geq \inf_{z\in G(x_0,\delta)} I_1(z) \geq (I_1(x_0) - \delta) \wedge \frac{1}{\delta}.$$

The large deviations principle for $(X^\varepsilon)_{\varepsilon>0}$ implies that

$$\begin{aligned}
-\inf_{z\in clF(x_0,\delta)} I_1(z) &\geq \limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in F(x_0,\delta)) \\
&\geq \liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in F(x_0,\delta)) \geq -\inf_{z\in F(x_0,\delta)} I_2(z).
\end{aligned}$$

Hence, it follows

$$I_2(x_0) \geq \inf_{z\in F(x_0,\delta)} I_2(z) \geq \inf_{z\in clF(x_0,\delta)} I_1(z) \geq (I_1(x_0) - \delta) \wedge \frac{1}{\delta}.$$

Since $\delta > 0$ can be chosen arbitrarily, the assumption $I_1(x_0) > I_2(x_0)$ is contradicted. $\square$

Fix $\mathcal{S}$ a Hausdorff regular space. Usually it is difficult to infer a large deviations principle for a $\mathcal{S}$- valued family of random variables $(X^\varepsilon)_{\varepsilon>0}$ defined in some probability space. We present a weaker definition that in practice is easier to prove.
In what follows in the rest of this section we fix $b(\varepsilon) := \frac{\varepsilon}{a^2(\varepsilon)}$ for all $\varepsilon > 0$ with $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ some function such that $b(\varepsilon) \to 0$. We assume either $a(\varepsilon) \to 0$ as $\varepsilon \to 0$ or $a := 1$.

Definition D.1.2 (**Weak large deviations principle**). *Let $(X^\varepsilon)_{\varepsilon>0}$ be a family of $\mathcal{S}$-valued random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $I : \mathcal{S} \longrightarrow [0,\infty]$. We say that $(X^\varepsilon)_{\varepsilon>0}$ satisfies a weak large deviations principle with good rate function I if the following holds.*

i) For every compact set $K \subset \mathcal{S}$ we have

$$\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in K) \leq -\inf_{x\in K} I(x).$$

ii) For every open set $G \subset \mathcal{S}$ we have

$$\liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in G) \geq -\inf_{x\in G} I(x).$$

It is obvious that the large deviations principle implies the weak large deviations principle, but the implication does not hold in general in the opposite direction. We consider the following example from *Gentz (2003)*.

Example D.1.1. Let us consider the family of $\mathbb{R}$- random variables $(X^\varepsilon)_{\varepsilon>0}$ defined in some probability space with laws $\mathbb{P} \circ (X^\varepsilon)^{-1} := \delta_{\frac{1}{\varepsilon}}$, where $\delta_{\frac{1}{\varepsilon}}$ is the Dirac measure in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ centered in $\frac{1}{\varepsilon}$. Let us prove that $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle with speed $b(\varepsilon) = \varepsilon$. Fix $K \in \mathcal{B}(\mathbb{R})$ a compact set and $\varepsilon > 0$ small enough such that $\frac{1}{\varepsilon} \in K^c$ since K is bounded. Hence, the upper bound i) in **Definition D.1.2** holds for the good rate function $I := \infty$. The lower bound of the definition of large deviations principle given in **Definition 1.0.1** is automatically satisfied for any $A \in \mathcal{B}(\mathbb{R})$. On the other hand, if we choose $F := [1, \infty) = \mathrm{cl}F$, then we infer that

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \delta_{\frac{1}{\varepsilon}}(F) = 0 > -\infty = \inf_{x\in clF} I(x),$$

which contradicts the upper bound of the definition of large deviations principle given in **Definition 1.0.1**.

To conclude from a weak large deviations principle a (full) large deviations principle it is necessary to assume an extra condition for the laws of the family $(X^\varepsilon)_{\varepsilon>0}$, called *exponential tightness*.

Definition D.1.3 (Exponential tightness). *Let $(X^\varepsilon)_{\varepsilon>0}$ be a family of $\mathcal{S}$- random variables defined om probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*
*We say that the family $(X^\varepsilon)_{\varepsilon>0}$ is **exponentially tight** (or the laws $(\mathbb{P} \circ (X^\varepsilon)^{-1})_{\varepsilon>0}$ are exponentially tight) with speed $b(\varepsilon)$ if for every $a < \infty$ there exists a compact set $K_a \subset \mathcal{S}$ such that*

$$\limsup_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in K_a^c) < -a.$$

The exponential tightness condition can be used to derive a full large deviations principle from a weak large deviations principle in the following sense.

Proposition D.1.2. *Let $(X^\varepsilon)_{\varepsilon>0}$ be a family of $\mathcal{S}$- random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ exponentially tight with speed $b(\varepsilon)$ in the sense of **Definition D.1.3**. Let us fix $I : \mathcal{S} \to [0, \infty]$ a rate function. We have the following.*

1) The lower bound,

$$\limsup_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in K) \le -\inf_{x\in K} I(x) \quad \textit{for evey compact } K \subset \mathcal{S},$$

implies the lower bound for closed sets $F \in \mathcal{S}$.

ii) The upper bound ,

$$\liminf_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in G) \ge -\inf_{x\in G} I(x), \quad \textit{for every open set } G \subset \mathcal{S},$$

implies that I is a good rate function.

Proof. We start to prove i). Let $F \subset \mathcal{S}$ be a closed set and we fix $a < \infty$ such that $\inf\limits_{x \in F} I(x) \geq a$. Using the fact that the laws of $(X^\varepsilon)_{\varepsilon>0}$ are exponentially tight with speed $b(\varepsilon)$ we choose $K_a \subset \mathcal{S}$ compact such that

$$\limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in K_a) < -a.$$

Then for every $\varepsilon > 0$ it follows

$$\mathbb{P}(X^\varepsilon \in F) \leq \mathbb{P}(X^\varepsilon \in F \cap K_a) + \mathbb{P}(X^\varepsilon \in K_a^c).$$

We observe that for every $x, y > 0$,

$$\ln(x+y) \leq ln(2x) \vee ln(2y) = \ln(x) \vee \ln(y) + ln2,$$

which implies that

$$\lim_{\varepsilon \to 0} b(\varepsilon) \ln(x+y) \leq \lim_{\varepsilon \to 0} b(\varepsilon)(\ln x \vee \ln y),$$

and therefore,

$$\begin{aligned}
\limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in F) &\leq \limsup_{\varepsilon \to 0} b(\varepsilon) \ln \Big(\mathbb{P}(X^\varepsilon \in F \cap K_a) + \mathbb{P}(X^\varepsilon \in F \cap K_a^c) \Big) \\
&\leq \limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in F \cap K_a) \vee \limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in K_a^c) \\
&= \limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in F \cap K_a) \\
&\leq - \inf_{x \in F \cap K_a} I(x) \\
&\leq -a.
\end{aligned}$$

We send $a \nearrow \inf\limits_{x \in F} I(x)$ and i) is proven.
We follow with the proof of the sentence ii). Fix $a < \infty$. We want to show that $I^{-1}([0, a])$ is compact. According to the definition of exponential tightness we fix a compact set $K_a \subset \mathcal{S}$ such that

$$\limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in K_a^c) \leq -a.$$

Applying the lower bound to the open set K_a^c it follows that

$$-\inf_{x \in K_a^c} I(x) \leq \liminf_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in K_a^c) < -a.$$

Consequently we have

$$\inf_{x \in K_a^c} I(x) > a.$$

This means that $I(x) \leq a$ implies that $x \in K_a$. The compactness of $I^{-1}([0, a])$ follows from the fact that it is a closed set (since I is lower semicontinuous) contained in K_a. $\square$

We discuss in what follows that if two families of $\mathcal{S}$-valued random variables $(X^\varepsilon)_{\varepsilon>0}$ and $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ defined on a probability space are asymptotically close in the sense of the next definition large deviations principles are indistinguishible.

Definition D.1.4 (Exponentially equivalence). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{S}, d)$ be a metric space equipped with its Borel σ-algebra $\mathcal{B}(\mathcal{S})$, for the topology given by the metric d. For every $\varepsilon > 0$, the laws of the families $(X^\varepsilon)_{\varepsilon>0}$ and $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ are asymptotically equivalent with speed $b(\varepsilon)$ if, for every $\delta > 0$, defining*

$$\Gamma_\delta := \{(x, y) \in \mathcal{S}^2 \mid d(x, y) > \delta\},$$

and supposing the measurability of $d(X^\varepsilon, \tilde{X}^\varepsilon)$, we have

$$\limsup_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}^\varepsilon(\Gamma_\delta) = \limsup_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(d(X^\varepsilon, \tilde{X}^\varepsilon) > \delta) = -\infty,$$

where $\mathbb{P}^\varepsilon := \mathbb{P} \circ (X^\varepsilon, \tilde{X}^\varepsilon)^{-1}$.

Theorem D.1.1 (Exponentially equivalent families preserve same large deviations principles). *Let $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ be a metric space with metric d and two families of $\mathcal{S}$- valued random variables $(X^\varepsilon)_{\varepsilon>0}$ and $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle with speed $b(\varepsilon)$ and good rate function I and that the laws of $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ are exponentially equivalent with speed $b(\varepsilon)$ to the laws of $(X^\varepsilon)_{\varepsilon>0}$. Then the family $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in $\mathcal{S}$ with good rate function I.*

Proof. 1. We start to prove that for any $x \in \mathcal{S}$ we have

$$I(x) = -\inf_{\delta>0} \limsup_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(\tilde{X}^\varepsilon \in B_\delta(x)) = -\inf_{\delta>0} \liminf_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(\tilde{X}^\varepsilon \in B_\delta(x)).$$

For this purpose let $\delta > 0$ and $x \in \mathcal{S}$ given. For every $\varepsilon > 0$ we have

$$\mathbb{P}(X^\varepsilon \in B_\delta(x)) \le \mathbb{P}(\tilde{X}^\varepsilon \in B_{2\delta}(x)) + \mathbb{P}^\varepsilon(\Gamma_\delta).$$

Using the lower bounds in the definition of large deviations principle presented in **Definition 1.0.1** for $(X^\varepsilon)_{\varepsilon>0}$ we conclude that

$$\begin{aligned}
-\inf_{z\in B_\delta(x)} I(z) &\le \liminf_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in B_\delta(x)) \\
&\le \liminf_{\varepsilon\to 0} b(\varepsilon) \ln \Big(\mathbb{P}(\tilde{X}^\varepsilon \in B_{2\delta}(x)) + \mathbb{P}^\varepsilon(\Gamma_\delta)\Big) \\
&\le \liminf_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(\tilde{X}^\varepsilon \in B_{2\delta}(x)) \vee \liminf_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}^\varepsilon(\Gamma_\delta).
\end{aligned}$$

Since the laws of $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ are exponentially equivalent to the laws of $(X^\varepsilon)_{\varepsilon>0}$, this implies

$$-\inf_{z\in B_\delta(x)} I(z) \le \liminf_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(\tilde{X}^\varepsilon \in B_{2\delta}(x)).$$

Reversing the roles of $(X^\varepsilon)_{\varepsilon>0}$ and $(\tilde{X}^\varepsilon)_{\varepsilon>0}$ and the same reasoning yields

$$-\inf_{z\in clB_{2\delta}(x)} I(z) \geq \limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in B_\delta(x)).$$

Noting that $\mathrm{cl}B_{2\delta}(x) \subset B_{3\delta}(x)$ it is implied that

$$\begin{aligned}-\inf_{z\in B_\delta(x)} I(z) &\leq \liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in B_{2\delta}(x))\\ &\leq \limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in B_{3\delta}(x))\\ &\leq -\inf_{z\in \mathrm{cl}B_{3\delta}(x)} I(z).\end{aligned}$$

Taking $\inf\limits_{\delta>0}$ on both sides of the previous inequalities proves the desired sentence.

2. Next we show that, given $x \in \mathcal{S}$ and $G \subset \mathcal{S}$ open such that $x \in G$, we have

$$-I(x) \leq \liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in G).$$

Using the previous sentence that was proven before, fixed $x \in G$, due to the fact that G is an open set, there exists $\delta > 0$ such that $B_\delta(x) \subset G$ and

$$-I(x) = \inf_{\delta>0}\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in B_\delta(x)) \leq \liminf_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in G),$$

which proves the desired estimate. We remark that this statement implies the lower bound of the large deviations principle with speed $b(\varepsilon)$ for $(\tilde{X}^\varepsilon)_{\varepsilon>0}$.

3. We continue with the proof, fixing now a closed set $F \subset \mathcal{S}$, $\delta > 0$ and writing $F^\delta := \{z \in \mathcal{S} \mid d(z, F) \leq \delta\}$. We show that

$$\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in F) \leq -\inf_{y\in F^\delta} I(y).$$

We note first that we have for every $\varepsilon > 0$

$$\mathbb{P}(\tilde{X}^\varepsilon \in F) \leq \mathbb{P}(X^\varepsilon \in F^\delta) + \mathbb{P}^\varepsilon(\Gamma_\delta).$$

Applying the upper bound from the definition of large deviations principle for $(X^\varepsilon)_{\varepsilon>0}$ we conclude

$$\begin{aligned}\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(\tilde{X}^\varepsilon \in F) &\leq \limsup_{\varepsilon\to 0} b(\varepsilon)\ln\Big(\mathbb{P}(X^\varepsilon \in F^\delta) + \mathbb{P}^\varepsilon(\Gamma_\delta)\Big)\\ &\leq \limsup_{\varepsilon\to} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in F^\delta) \vee \limsup_{\varepsilon\to} b(\varepsilon)\ln \mathbb{P}^\varepsilon(\Gamma^\delta)\\ &\leq -\inf_{y\in F^\delta} I(y) \vee \limsup_{\varepsilon\to} b(\varepsilon)\ln \mathbb{P}^\varepsilon(\Gamma^\delta).\end{aligned}$$

Using the exponentially equivalence of $(X^\varepsilon)_{\varepsilon>0}$ and $(\tilde{X}^\varepsilon)_{\varepsilon>0}$, we conclude that

$$\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{y\in F^\delta} I(y).$$

4. Given $F \subset \mathcal{S}$ a closed set we show that

$$\inf_{y \in F} I(y) = \lim_{\delta \to 0} \inf_{y \in F^\delta} I(y),$$

which together with the sentence 3. finishes the proof of the upper bound of the large deviations principle for the family of random variables $(\tilde{X}^\varepsilon)_{\varepsilon>0}$. Let $a > 0$. In what follows we show the following,

$$\lim_{\delta \to 0} \inf_{y \in F^\delta} I(y) \geq \inf_{y \in F} I(y) - a.$$

Without loss of generality, assume that $\lim_{\delta \to 0} \inf_{y \in F^\delta} I(y) < \infty$.
Let $b := \lim_{\delta \to 0} \inf_{y \in F^\delta} I(y) + a$. Then for every $\delta > 0$ we have $\inf_{y \in F^\delta} I(y) \leq b$. From the definition

$$F^\delta \cap I^{-1}([0, b]) \neq \emptyset.$$

We observe that $F^\delta \cap I^{-1}([0, b])$ is a compact set, due to the fact I is a good rate function. We can write

$$F \cap I^{-1}([0, b]) := \bigcap_{\delta > 0} \left(F^\delta \cap I^{-1}([0, b]) \right) \neq \emptyset.$$

The last expression implies that

$$\inf_{y \in F} I(y) \leq b,$$

which proves the desired inequality stated in the beginning of point 4. Therefore the proof is complete. $\square$

D.2 The contraction principle

Let us fix $b(\varepsilon) := \frac{\varepsilon}{a^2(\varepsilon)}$ for some function $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $b(\varepsilon) \to 0$. We allow the two different cases:

i) $a(\varepsilon) \to 0$ as $\varepsilon \to 0$ or

ii) $a(\varepsilon) := 1$ for every $\varepsilon > 0$.

Theorem D.2.1 (**Contraction principle**). *Let $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ and $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ be topological spaces equiped with the respective Borel σ-algebras and $f : \mathcal{S} \longrightarrow \mathcal{T}$ a continuous mapping. Fix $I : \mathcal{S} \longrightarrow [0, \infty]$ a good rate function.*

i) We define the functional

$$\tilde{I} : \mathcal{T} \longrightarrow [0, \infty]$$
$$\tilde{I}(y) := \inf\{I(x) \mid x \in \mathcal{S} \text{ such that } y = f(x)\}.$$

Therefore we conclude that $\tilde{I}$ is a good rate function on $\mathcal{T}$.

ii) Let $(X^\varepsilon)_{\varepsilon>0}$ be a family of $\mathcal{S}$- valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ obeying a large deviations principle with speed $b(\varepsilon)$ and good rate function I. Then the family $(Y^\varepsilon)_{\varepsilon>0} := (f(X^\varepsilon))$ of $\mathcal{T}$- valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies a large deviations principle with same speed $b(\varepsilon)$ with respect to the good rate function $\tilde{I}$.

Proof. We prove that $\tilde{I}$ is a good rate function. Fix $a < \infty$. In what follows we show the compacity of

$$\{y \in \mathcal{T} : \tilde{I}(y) \leq a\}.$$

We observe that $\{y \in \mathcal{T} : \tilde{I}(y) \leq a\} = f(I^{-1}([0, a]))$ is the image by f of the compact set $I^{-1}([0, a])$, since I is a good rate function. Due to the fact that f is continuous the compacity of $\{y \in \mathcal{T} : \tilde{I}(y) \leq a\}$ follows.

We prove the upper and lower bounds of the large deviations principle stated in (D.1.2). Let $G \subset \mathcal{T}$ be an open set. Then $f^{-1}(G) \subset \mathcal{S}$ is an open set by continuity of f and this implies

$$\begin{aligned}\liminf_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(Y^\varepsilon \in G) &= \liminf_{\varepsilon\to 0} b(\varepsilon)\mathbb{P}(X^\varepsilon \in f^{-1}(G)) \\ &\geq - \inf_{x\in f^{-1}(G)} I(x) \\ &= - \inf_{y\in G} \tilde{I}(y).\end{aligned}$$

Let now $F \subset \mathcal{T}$ be a closed set. Then $f^{-1}(F) \subset \mathcal{S}$ is a closed set by continuity of f. Therefore, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{P}(Y^\varepsilon \in F) &= \limsup_{\varepsilon\to 0} b(\varepsilon)\mathbb{P}(X^\varepsilon \in f^{-1}(F)) \\ &\leq - \inf_{x\in f^{-1}(F)} I(x) \\ &= - \inf_{y\in F} \tilde{I}(y).\end{aligned}$$

Hence we conclude that the family $(y^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle with speed $b(\varepsilon)$ in the space $\mathcal{T}$ with good rate function $\tilde{I}$. $\square$

We use in **Chapter 4** the more general form of the contraction principle. Its proof is a consequence of **Proposition D.1.1** and we refer the reader to **Theorem 4.2.23** in *Dembo and Zeitoni (1998)* for a proof.

Theorem D.2.2 (**Extended contraction Principle**). *Let $f : \mathcal{S} \to Y$ be a continuous mapping from a topological vector space $\mathcal{S}$ to a certain metric space $(\mathcal{T}, d)$ and $(X^\varepsilon)_{\varepsilon>0}$ be a family of random variables defined in a common probabiliy space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathcal{S}$,satisfying a large deviations principle with a good rate function $I : X \to [0, \infty]$.*
For every $\varepsilon > 0$ let $f^\varepsilon : \mathcal{S} \to \mathcal{T}$ be a continuous functions and let us assume that there exists a measurable map $f : \mathcal{S} \longrightarrow \mathcal{T}$ such that for every $\alpha < \infty$,

$$\limsup_{\varepsilon\to 0} \sup_{\{x \ \mid \ I(x)\leq\alpha\}} d(f^\varepsilon(x), f(x)) = 0.$$

Then $(Y^\varepsilon)_{\varepsilon>0} := (f^\varepsilon(X^\varepsilon))_{\varepsilon>0}$ satisfies a large deviations principle with good rate function

$$\tilde{I}(y) = \inf\{I(x) : x \in X \text{ and } f(x) = y\}.$$

D.3 The Laplace-Varadhan Principle

Let $\mathcal{S}$ be a Polish space, i.e. a complete separable space equipped with some metric d and let us consider a family of $\mathcal{S}$- valued random variables $(X^\varepsilon)_{\varepsilon>0}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expectation operator is denoted by $\mathbb{E}$.
We fix, such as in the previous section, $b(\varepsilon) := \frac{\varepsilon}{a^2(\varepsilon)}$ for some function $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $b(\varepsilon) \to 0$, allowing two different cases:

i) $a(\varepsilon) \to 0$ as $\varepsilon \to 0$ or

ii) $a(\varepsilon) := 1$ for every $\varepsilon > 0$.

In this section we present an equivalence result between the large deviations principle stated in **Definition 1.0.1** and the *Laplace-Varadhan principle* that we state below.

Definition D.3.1 (**Laplace Principle**). *Let $\mathcal{S}$ be a Polish space. A family $(X^\varepsilon)_{\varepsilon>0}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathcal{S}$ satisfies the Laplace Principle with speed $b(\varepsilon)$ and good rate function $I : \mathcal{S} \longrightarrow [0, \infty]$ if:*

- *I is a good rate function and*
- *for all continuous bounded functions defined in $\mathcal{S}$, $h \in C_b(\mathcal{S})$, the following property holds:*

$$\lim_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{E}\Big[\exp\Big(-\frac{1}{b(\varepsilon)} h(X^\varepsilon)\Big)\Big] = -\inf_{x\in\mathcal{S}}\{h(x) + I(x)\}.$$

The next theorem is due to Varadhan (1966) and it links the large deviations principle and the Laplace-Varadham principle.

Theorem D.3.1 (**Varadhan 1966**). *If $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in $\mathcal{S}$ with speed $b(\varepsilon)$ and good rate function I, then $(X^\varepsilon)_{\varepsilon>0}$ satisfies the Laplace-Varadhan principle with the same speed $b(\varepsilon)$ and the same good rate function I.*

Proof. 1. Fix $h \in C_b(\mathcal{S})$. We prove the following upper bound,

$$\limsup_{\varepsilon\to 0} b(\varepsilon) \ln \mathbb{E}\Big[\exp\Big(-\frac{1}{b(\varepsilon)} h(X^\varepsilon)\Big)\Big] \le -\inf_{x\in\mathcal{S}}\{h(x) + I(x)\}.$$

Let $C := ||h||_\infty < \infty$ since $h \in C_b(\mathcal{S})$. Given $k \in \mathbb{N}$ and $j \in \{1, ..., 2k\}$ we consider the closed sets

$$F_j^k := \Big\{x \in \mathcal{S} \mid -C + (j-1)\frac{C}{k} \le -h(x) \le -C + j\frac{C}{k}\Big\}.$$

Using the upper bound of the large deviations of $(X^\varepsilon)_{\varepsilon>0}$ and the definition of F_j^k, we conclude that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} b(\varepsilon)\ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}h(X^\varepsilon)}\Big] \\
&= \limsup_{\varepsilon\to 0} b(\varepsilon)\ln\Big(\sum_{j=1}^{2k}\int_{F_j^k} e^{-\frac{1}{b(\varepsilon)}h(x)}\mathbb{P}(X^\varepsilon\in dx)\Big) \\
&\le \limsup_{\varepsilon\to 0} b(\varepsilon)\ln\Big(\sum_{j=1}^{2k} e^{\frac{1}{b(\varepsilon)}(-C+\frac{jC}{k})}\mathbb{P}(X^\varepsilon\in F_j^k)\Big) \\
&\le \limsup_{\varepsilon\to 0} b(\varepsilon)\ln\Big(2k\max_{j=1,\dots,2k}\Big\{e^{\frac{1}{b(\varepsilon)}(-C+\frac{jC}{k})}\Big\}\mathbb{P}(X^\varepsilon\in F_j^k)\Big) \\
&= \limsup_{\varepsilon\to 0} b(\varepsilon)\Big(\max_{j=1,\dots,2k}\Big\{-\frac{C}{b(\varepsilon)}+\frac{jC}{kb(\varepsilon)}\Big\}+\ln\mathbb{P}(X^\varepsilon\in F_j^k)\Big) \\
&\le \max_{j=1,\dots,2k}\sup_{x\in F_j^k}\Big\{-C+\frac{jC}{k}-I(x)\Big\} \\
&\le \max_{j=1,\dots,2k}\sup_{x\in F_j^k}\{-h(x)-I(x)\}+\frac{C}{k} \\
&\le \sup_{x\in\mathcal{S}}\{-h(x)-I(x)\}+\frac{C}{k}.
\end{aligned}$$

We obtain the desired upper bound sending $k\to\infty$.

2. Fix $h\in C_b(\mathcal{S})$. We prove the following lower bound

$$\liminf_{\varepsilon\to 0} b(\varepsilon)\log\mathbb{E}\Big[\exp\Big(-\frac{1}{b(\varepsilon)}h(X^\varepsilon)\Big)\Big]\ge -\inf_{x\in\mathcal{S}}\{h(x)+I(x)\}.$$

Fixed $x\in\mathcal{S}$ and given $\delta>0$ we consider the open set (due to the continuity of h),

$$G:=\{y\in\mathcal{S}\mid h(y)<h(x)+\delta\}.$$

Using the lower bound in the definition of large deviations principle for $(X^\varepsilon)_{\varepsilon>0}$ we obtain the following estimate

$$\begin{aligned}
\liminf_{\varepsilon\to 0} b(\varepsilon)\ln\mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}h(X^\varepsilon)}\Big] &\ge \liminf_{\varepsilon\to 0} b(\varepsilon)\ln\mathbb{E}\Big[\mathbf{1}_G(X^\varepsilon)e^{-\frac{1}{b(\varepsilon)}h(X^\varepsilon)}\Big] \\
&\ge \liminf_{\varepsilon\to 0} b(\varepsilon)\ln\mathbb{E}\Big[1_G(X^\varepsilon)e^{-\frac{1}{b(\varepsilon)}(h(x)+\delta)}\Big] \\
&= \liminf_{\varepsilon\to 0} b(\varepsilon)\Big(\ln\mathbb{E}[\mathbf{1}_G(X^\varepsilon)]-\frac{1}{b(\varepsilon)}(h(x)+\delta)\Big) \\
&\ge -\inf_{z\in G} I(z)-h(x)-\delta \\
&\ge -(I(x)+h(x))-\delta \\
&\ge -\inf_{x\in\mathcal{S}}(h(x)+I(x))-\delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary the result follows.

$\square$

We present the converse of Varadhan's result in the following sense.

Theorem D.3.2 (Laplace-Varadhan principle implies a large deviations principle). *If I is a good rate function on $\mathcal{S}$ and for every $h \in C_b(\mathcal{S})$ the following limit holds*

$$\lim_{\varepsilon \to 0} b(\varepsilon)\mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}h(X^\varepsilon)}\Big] = -\inf_{x \in \mathcal{S}}(h(x) + I(x)),$$

then $(X^\varepsilon)_\varepsilon$ satisfies a large deviations principle with speed $b(\varepsilon)$ and with good rate function I.

Proof. 1. We prove the upper bound for $(X^\varepsilon)_{\varepsilon>0}$. Given $F \in \mathcal{B}(\mathcal{S})$ a closed set we want to show that

$$\limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x \in F} I(x).$$

Fix $F \in \mathcal{B}(\mathcal{S})$ a closed set and define the lower semicontinuous function

$$f(x) := \begin{cases} 0 & \text{if } x \in F \\ \infty & \text{if } x \in F^c. \end{cases}$$

For $k \in \mathbb{N}$ we define

$$f_k(x) := k(d(x, F) \wedge 1),$$

where $d(x, F)$ is the distance between the point x and the closed set F. By construction f_k is a bounded continuous function and $f_k \nearrow f$ as $k \to \infty$. Then it follows

$$\begin{aligned} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in F) &= b(\varepsilon) \ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}f(X^\varepsilon)}\Big] \\ &\leq b(\varepsilon) \ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}f_k(X^\varepsilon)}\Big]. \end{aligned}$$

Hence, using the *Laplace-Varadhan principle* applied to f_k, we conclude that

$$\begin{aligned} \limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in F) &\leq b(\varepsilon) \ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}f_k(X^\varepsilon)}\Big] \\ &= -\inf_{x \in \mathcal{S}}\{f_k(x) + I(x)\}. \end{aligned}$$

The proof of the upper bound is finished if we prove

$$-\lim_{k \to \infty} \inf_{x \in \mathcal{S}}\{f_k(x) + I(x)\} = -\inf_{x \in F} I(x).$$

The relation $f_k \leq f$ and the definition of the function f, that implies $\inf_{x \in \mathcal{S}} f(x) = \inf_{x \in F} = 0$, yield

$$\begin{aligned}\inf_{x \in \mathcal{S}} \{f_k(x) + I(x)\} &\leq \inf_{x \in \mathcal{S}} \{f(x) + I(x)\} \\ &= \inf_{x \in F} \{I(x) + f(x)\} \\ &= \inf_{x \in F} I(x).\end{aligned}$$

In what follows we show the reverse inequality

$$\lim_{k \to \infty} \inf_{x \in \mathcal{S}} \{f_k(x) + I(x)\} \geq - \inf_{x \in F} I(x).$$

The case $- \inf_{x \in F} I(x) = 0$ is trivial. Let us assume that

$$0 < \inf_{x \in F} I(x) < \infty.$$

The relations

$$\begin{aligned}\inf_{x \in \mathcal{S}} \{f_k(x) + I(x)\} &= \inf_{x \in F} \{f_k(x) + I(x)\} \wedge \inf_{x \in F^c} \{f_k(x) + I(x)\} \\ &= \inf_{x \in F} I(x) \wedge \inf_{x \in F^c} \{f_k(x) + I(x)\}\end{aligned}$$

show that it is enough to prove

$$\lim_{k \to \infty} \inf_{x \in F^c} \{f_k(x) + I(x)\} \geq \inf_{x \in F} I(x). \tag{D.3.1}$$

Let us argue by contradiction. Assume that

$$\lim_{k \to \infty} \inf_{x \in F^c} \{f_k(x) + I(x)\} < \inf_{x \in F} I(x).$$

Choose $\delta > 0$ such that

$$\lim_{k \to \infty} \inf_{x \in F^c} \{f_k(x) + I(x)\} < \inf_{x \in F} I(x) - 2\delta.$$

For the sake of simplicity in the notation let $(f_k)_{k \in \mathbb{N}}$ be a subsequence of the sequence $(f_k)_{k \in \mathbb{N}}$ itself such that

$$\inf_{x \in F^c} \{f_k(x) + I(x)\} \leq \inf_{x \in F} I(x) - \delta.$$

For each $k \in \mathbb{N}$, let $x_k \in F^c$ such that

$$f_k(x_k) + I(x_k) \leq \inf_{x \in F} I(x) - \delta. \tag{D.3.2}$$

By construction of f_k the inequality before implies that

$$d(x_k, F) \to 0 \quad \text{as } k \to \infty.$$

In particular there exists $y_k \in F$ such that $d(x_k, y_k) \to 0$ as $k \to \infty$.
The inequality (D.3.2) implies that

$$\sup_{k\in\mathbb{N}} I(x_k) \le \inf_{x\in F} I(x) - \delta.$$

Since I is a good rate function its sublevel sets are compacts. Hence, there exists a subsequence of $(x_k)_{k\in\mathbb{N}}$ that we denote $(x_k)_{k\in\mathbb{N}}$, for sake of simplicity in the notation, and a point

$$\bar{x} \in \{x \in \mathcal{S} \mid I(x) \le \inf_{y\in F} I(y) - \delta\}, \tag{D.3.3}$$

such that $d(x_k, \bar{x}) \to 0$ as $k \to \infty$. Since $F \subset \mathcal{S}$ is a closed set, it follows that $\bar{x} \in F$. Hence $I(\bar{x}) \ge \inf_{y\in F} I(y)$, which contradicts (D.3.3). The proof of (D.3.1) is complete.
Let us assume now that $\inf_{x\in F} I(x) = \infty$. Then (D.3.1) reads as

$$\liminf_{k\to\infty} \inf_{x\in F^c} \{f_k(x) + I(x)\} = \infty.$$

We prove the statement above by contradiction. Assume that there exists $C > 0$ such that

$$\liminf_{k\to\infty} \inf_{x\in F^c} \{f_k(x) + I(x)\} < C.$$

Following the strategy of the proof of (D.3.1), with $\inf_{x\in F} I(x)$ replaced by C, we end up with a contradiction again. Since $C > 0$ was fixed arbitrarily, the proof of

$$\liminf_{k\to\infty} \inf_{x\in F^c} \{f_k(x) + I(x)\} = \infty.$$

is finished.

2. We prove the lower bound of the large deviations principle for $(X^\varepsilon)_{\varepsilon>0}$. Given $G \subset \mathcal{S}$ open, we show

$$\liminf_{\varepsilon\to 0} b(\varepsilon)\mathbb{P}(X^\varepsilon \in G) \ge -\inf_{x\in G} I(x).$$

If $\inf_{x\in G} I(x) = \infty$, nothing has to be proven. Let us assume $\inf_{x\in G} I(x) < \infty$. We fix a point $x \in G$ such that $I(x) < \infty$. Choose $C > I(x)$ and $\delta > 0$ such that the open δ-ball $B_\delta(x)$ centered in x is contained in G. We define the auxiliary function

$$f(y) := C\Big(\frac{d(x,y)}{\delta} \wedge 1\Big).$$

By construction $f \in C_b(\mathcal{S})$ and the following bounds hold

$$0 \leq f(z) \leq C.$$

By construction $f(x) = 0$ on $B_\delta(x)$ and $f \equiv C$ on $B_\delta^c(x)$. It follows that

$$\begin{aligned}
\mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}f(X^\varepsilon)}\Big] &\leq e^{-\frac{1}{b(\varepsilon)}C}\mathbb{P}(X^\varepsilon \in B_\delta^c(x)) + \mathbb{P}(X^\varepsilon \in B_\delta(x)) \\
&\leq e^{-\frac{1}{b(\varepsilon)}C} + \mathbb{P}(X^\varepsilon \in B_\delta(x)) \\
&\leq 2\Big(e^{-\frac{1}{b(\varepsilon)}C} \vee \mathbb{P}(X^\varepsilon \in B_\delta(x))\Big).
\end{aligned}$$

Taking the logarithm in the previous estimate we have

$$\ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}h(X^\varepsilon)}\Big] \leq \ln 2 + (-\frac{1}{b(\varepsilon)}C) \vee \ln \mathbb{P}(X^\varepsilon \in B_\delta(x)).$$

Using the definition of the function f we conclude

$$\begin{aligned}
-C \vee \liminf_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in B_\delta(x)) &\geq \liminf_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}h(X^\varepsilon)}\Big] \\
&\geq -\inf_{y \in \mathcal{S}}\{f(y) + I(y)\} \\
&\geq -f(x) - I(x) \\
&= -I(x).
\end{aligned}$$

Since $C > I(x)$ and $B_\delta(x) \subset G$, it follows

$$\begin{aligned}
\liminf_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in G) &\geq \liminf_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{P}(X^\varepsilon \in B_\delta(x)) \\
&\geq -I(x) \\
&\geq -\inf_{y \in G} I(y),
\end{aligned}$$

which finishes the proof. □

D.4 The relative entropy and properties

In what follows $\mathcal{S}$ is a Polish space and $\mathcal{P}(\mathcal{S})$ is the space of the probability measures defined in $\mathcal{S}$.

Definition D.4.1 (Relative entropy). Let $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{S})$. *The relative entropy of* $\mathbb{P}$ *with respect to* $\mathbb{Q}$ *is the functional*

$$R(.||.) : \mathcal{P}(\mathcal{S}) \times \mathcal{P}(\mathcal{S}) \longrightarrow [0, \infty]$$

$$R(\mathbb{P}||\mathbb{Q}) := \begin{cases} \displaystyle\int_{\mathcal{S}} \ln\Big(\frac{d\mathbb{P}}{d\mathbb{Q}}(x)\Big)\mathbb{P}(dx), & \textit{if } \mathbb{P} \ll \mathbb{Q} \\ \infty, & \textit{if else }. \end{cases}$$

Remark D.4.1. *Intuitively the relative entropy between two probability measures is a measure of the information gained when one revises ones beliefs from the prior probability distribution to the posterior probability measure. This concept is intimately related with the second law of thermodynamics . The reader will find in the book Ellis (1985) a clear illustrations of the interplay between large deviations theory, thermodynamics and particle systems through the use of the concept of relative entropy. The work Ellis (1999) is a panoramic survey of literature and historical developments of large deviations where the use of relative entropy is explained through examples from thermodynamics. In the seminal work Boltzmann (1877), in a modern terminology, it is used the notion of relative entropy through large deviations asymptotics for multinomial probabilities, in order to derive properties of certain gases systems related to the fundamental law of thermodynamics. We account the classical works Kullback and Leibler (1951) Kullback (1997) for illustrations in information theory of the concept of relative entropy (called also directed divergence).*

Remark D.4.2. [Immediate conclusions from the definition of relative entropy.] *If* $\mathbb{P}$ *and* $\mathbb{Q}$ *are probability measures in* $\mathcal{S}$ *such that* $\mathbb{P} \ll \mathbb{Q}$*, due to Radon-Nykodym theorem there exists a density* $f := \frac{d\mathbb{P}}{d\mathbb{Q}}$ *in* $L^1(\mathcal{S}, \mathbb{Q})$ *uniquely determined* $\mathbb{Q}$*- a.s. Then*

$$R(\mathbb{P}||\mathbb{Q}) = \int_{\mathcal{S}} f(x) \ln f(x) \mathbb{Q}(dx).$$

We note that $\lim\limits_{x \to 0^+} x \ln x = 0$*, which makes the integral above well-defined.*
Since $\int_{\mathcal{S}} f(x)\mathbb{Q}(dx) = 1$ *and* $x \ln x \geq x - 1$ *for all* $x \geq 0$*, with equality if and only if* $x = 1$*, we have that*

$$R(\mathbb{P}||\mathbb{Q}) \geq 0 \quad \textit{and } R(\mathbb{P}||\mathbb{Q}) = 0 \textit{ if and only if } \mathbb{P} = \mathbb{Q}.$$

In what follows we list a set of basic properties of relative entropy that will be useful in the sequel. We refer the reader to *Dupuis and Ellis (1997)*- pp 29-30 for a proof.

Lemma D.4.1 (**Basic properties of relative entropy**). *We have the following properties.*

1. *$R(.||.) : \mathcal{P}(\mathcal{S}) \times \mathcal{P}(\mathcal{S}) \longrightarrow [0, \infty]$ is a non-negative, convex, lower semicontinuous functional.*

2. *For every $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$ $R(.||\mathbb{Q})$ is strictly convex on*
$$\{\mathbb{P} \in \mathcal{P}(\mathcal{S}) \mid R(\mathbb{P}||\mathbb{Q}) < \infty\}.$$

3. *For every measure $\mathbb{P} \in \mathcal{P}(\mathcal{S})$, $R(.||\mathbb{P})$ has compact sublevel sets.*

In the next section we use the following result, known as the *contraction property of relative entropy.* We refer the reader to *Kullback and Leibler (1951)*-**Theorem 4.1** or to *Dupuis and Ellis (1997)*- **Lemma E.2.1** for a proof.

Lemma D.4.2 (**Contraction property**). *Let $\mathcal{S}, \mathcal{T}$ be Polish spaces and $\psi : \mathcal{T} \longrightarrow \mathcal{S}$ be a Borel measurable map. If $\mathbb{P} \in \mathcal{P}(\mathcal{S})$ and $\mathbb{Q} \in \mathcal{P}(\mathcal{T})$, then*
$$R(\mathbb{P}||\mathbb{Q} \circ \psi^{-1}) = \inf_{\gamma \in \mathcal{P}(\mathcal{T}): \gamma \circ \psi^{-1} = \mathbb{P}} R(\gamma||\mathbb{Q}).$$

The following theorem contains the first variational formula of the so called Laplace functionals in terms of the relative entropy. This theorem opens the way to the characterization of functionals of families of random variables in terms of the relative entropy. This principle is exploited in the next section, in order to derive the variational formula for functionals of Poisson random measures.

Theorem D.4.1 (**Variational formula for Laplace functionals in terms of relative entropy**). *Let $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$. Then for all bounded measurable function $g \in M_b(\mathcal{S})$, we have*
$$-\ln \int_{\mathcal{S}} e^{-g(x)} \mathbb{Q}(dx) = \inf_{\mathbb{P} \in \mathcal{P}(\mathcal{S})} \Big\{ R(\mathbb{P}||\mathbb{Q}) + \int_{\mathcal{S}} g(x) \mathbb{P}(dx) \Big\}.$$
The infimum in the variational formula above is attained at some $\mathbb{P}^* \in \mathcal{P}(\mathcal{S})$ such that $\mathbb{P}^* \ll \mathbb{Q}$ with density given $\mathbb{Q}$- a.s. by
$$\frac{d\mathbb{P}^*}{d\mathbb{Q}}(x) := \frac{e^{-g(x)}}{\int_{\mathcal{S}} e^{-g(y)} \mathbb{Q}(dy)}, \quad x \in \mathcal{S}\ \mathbb{Q}\text{-}a.s\ .$$

Proof. Let $g \in M_b(\mathcal{S})$ and define $\mathbb{P}^* \in \mathcal{P}(\mathcal{S})$ with a density with respect to $\mathbb{Q}$ given by
$$\frac{d\mathbb{P}^*}{d\mathbb{Q}}(x) := \frac{e^{-g(x)}}{\int_{\mathcal{S}} e^{-g(y)} \mathbb{Q}(dy)}, \quad x \in \mathcal{S}\ \mathbb{Q}\text{-a.s.}$$
By definition of the density of $\mathbb{P}^*$ with respect to $\mathbb{Q}$ we observe that $\mathbb{P}^*$ and $\mathbb{Q}$ are mutually absolutely continuous. Fix $\mathbb{P} \in \mathcal{P}(\mathcal{S})$ be such that $R(\mathbb{P}||\mathbb{Q}) < \infty$. Then $\mathbb{P}$ is absolutely

continuous with respect to $\mathbb{Q}$ with some density $\frac{d\mathbb{P}}{d\mathbb{Q}}$ determined $\mathbb{Q}$- a.s.
It follows that $\mathbb{P}$ is also absolutely continuous with respect to $\mathbb{P}^*$ with density

$$\frac{d\mathbb{P}}{d\mathbb{P}^*} = \frac{d\mathbb{P}}{d\mathbb{Q}}\frac{d\mathbb{Q}}{d\mathbb{P}^*} \quad \text{with } \frac{d\mathbb{Q}}{d\mathbb{P}^*} = \frac{e^g}{\int_{\mathcal{S}} e^g d\mathbb{P}^*}.$$

Hence,

$$\begin{aligned} R(\mathbb{P}||\mathbb{Q}) + \int_{\mathcal{S}} g d\mathbb{P} &= \int_{\mathcal{S}} \ln\Big(\frac{d\mathbb{P}}{d\mathbb{Q}}\Big) d\mathbb{P} + \int_{\mathcal{S}} g d\mathbb{P} \\ &= \int_{\mathcal{S}} \ln\Big(\frac{d\mathbb{P}}{d\mathbb{P}^*}\Big) d\mathbb{P} + \int_{\mathcal{S}} \ln\Big(\frac{d\mathbb{P}^*}{d\mathbb{Q}}\Big) d\mathbb{P} + \int_{\mathcal{S}} g d\mathbb{P} \\ &= R(\mathbb{P}||\mathbb{P}^*) + \int_{\mathcal{S}} \ln e^{-g} d\mathbb{P} - \int_{\mathcal{S}} \ln\Big(\int_{\mathcal{S}} e^{-g} d\mathbb{Q}\Big) d\mathbb{P} + \int_{\mathcal{S}} g d\mathbb{P} \\ &= R(\mathbb{P}||\mathbb{P}^*) - \ln \int_{\mathcal{S}} e^{-g} d\mathbb{Q}. \end{aligned}$$

Since $R(\mathbb{P}^*||\mathbb{P}) \geq 0$, with $R(\mathbb{P}^*||\mathbb{P}) = 0$ if and only if $\mathbb{P} = \mathbb{P}^*$, due to **Remark D.4.2**, we conclude the result. □

Theorem D.4.2 (**Donsker-Varadhan theorem**). *Let* $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{S})$. *Then*

$$R(\mathbb{P}||\mathbb{Q}) = \sup_{g \in M_b(\mathcal{S})} \Big\{ \int_{\mathcal{S}} g(x)\mathbb{P}(dx) - \ln \int_{\mathcal{S}} e^{g(x)}\mathbb{Q}(dx) \Big\}.$$

Proof. Let $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{S})$. Using the previous result **Theorem D.4.1** we have, for all $g \in M_b(\mathcal{S})$,

$$R(\mathbb{P}||\mathbb{Q}) \geq -\int_{\mathcal{S}} g d\mathbb{P} - \ln \int_{\mathcal{S}} e^{-g} d\mathbb{Q}$$

which yields

$$\begin{aligned} R(\mathbb{P}||\mathbb{Q}) &\geq \sup_{g \in M_b(\mathcal{S})} \Big\{ -\int_{\mathcal{S}} g d\mathbb{P} - \ln \int_{\mathcal{S}} e^{-g} d\mathbb{Q} \Big\} \\ &= \sup_{g \in M_b(\mathcal{S})} \Big\{ \int_{\mathcal{S}} g d\mathbb{P} - \ln \int_{\mathcal{S}} e^{g} d\mathbb{Q} \Big\}. \end{aligned}$$

Let us define the functional

$$F : M_b(\mathcal{S}) \longrightarrow \mathbb{R}$$

$$F(g) := \int_{\mathcal{S}} g d\mathbb{P} - \ln \int_{\mathcal{S}} e^g d\mathbb{Q}.$$

The inequalities above imply that $R(\mathbb{P}||\mathbb{Q}) \geq \sup_{g \in M_b(\mathcal{S})} F(g)$. We follow with the proof of the equality. It is enought to argue that there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset M_b(\mathcal{S})$ such that

$$\limsup_{n\to\infty} F(g_n) = R(\mathbb{P}||\mathbb{Q}).$$

1. Let us first assume that $\mathbb{P}$ is not absolutely continuous with respect to $\mathbb{Q}$. Then $R(\mathbb{P}||\mathbb{Q}) = \infty$ and there exists $A \in \mathcal{B}(\mathcal{S})$ such that $\mathbb{P}(A) > 0$ and $\mathbb{Q}(A) = 0$. We choose a set A like that and we define $g_n := n\mathbf{1}_A$. For every $n \in \mathbb{N}$, $g_n = 0$ $\mathbb{Q}$- a.s. It follows that

$$\int_{\mathcal{S}} e^{g_n} d\mathbb{Q} = 1 \quad \text{which implies } \ln \int_{\mathcal{S}} e^{g_n} d\mathbb{Q} = 0.$$

Hence,

$$\limsup_{n\to\infty} F(g_n) = \limsup_{n\to\infty} \int_{\mathcal{S}} g_n d\mathbb{P} = \limsup_{n\to\infty} n\mathbb{P}(A) = \infty.$$

2. Let us suppose that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$. We choose a measurable L^1 integrable $\mathbb{Q}$-a.s. version of the Radon-Nykodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$ that we denote by $f : \mathcal{S} \longrightarrow [0, \infty)$. Then, it follows that

$$R(\mathbb{P}||\mathbb{Q}) = \int_{\mathcal{S}} f(x) \ln f(x) \mathbb{Q}(dx) \in [0, \infty].$$

We define a sequence $(g_n)_{n\in\mathbb{N}} \subset M_b(\mathcal{S})$ as follows,

$$g_n(x) := \ln f(x)\mathbf{1}_{[\frac{1}{n},n]}(f(x)) - n\mathbf{1}_{\{0\}}(f(x)) \quad \text{for } x \in \mathcal{S}.$$

It is implied that

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathcal{S}} g_n d\mathbb{P} &= \lim_{n\to\infty} \int_{\mathcal{S}} f \ln f \cdot \mathbf{1}_{[\frac{1}{n},n]}(f) d\mathbb{Q} \\
&= \lim_{n\to\infty} \Big(\int_{\mathcal{S}} \Big(f \ln f + \mathbf{1}_{(0,\infty)} \Big) \mathbf{1}_{[\frac{1}{n},n]}(f) d\mathbb{Q} - \int_{\mathcal{S}} \mathbf{1}_{[\frac{1}{n},n]}(f) d\mathbb{Q} \Big) \\
&= \int_{\mathcal{S}} f \ln f d\mathbb{Q} + \mathbb{Q}(f > 0) - \mathbb{Q}(f > 0) \\
&= R(\mathbb{P}||\mathbb{Q}),
\end{aligned}$$

where we used in the third line monotone convergence theorem.

Using dominated convergence theorem in the following estimate we conclude that

$$\begin{aligned}
&\lim_{n\to\infty} \ln \int_{\mathcal{S}} e^{g_n} d\mathbb{Q} \\
&= \ln \Big(\lim_{n\to\infty} \int_{\mathcal{S}} (f \cdot \mathbf{1}_{[\frac{1}{n},n]}(f) + \mathbf{1}_{(0,\frac{1}{n})\cup(n,\infty)}(f) + e^{-n}\mathbf{1}_{\{0\}}(f)) d\mathbb{Q} \Big) \\
&= \ln \int_{\mathcal{S}} f d\mathbb{Q} \\
&= \ln 1 \\
&= 0.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} F(g_n) = \lim_{n\to\infty} \int_{\mathcal{S}} g_n d\mathbb{P} = R(\mathbb{P}||\mathbb{Q}),$$

which concludes the proof. $\square$

The variational formula of the Donsker-Varadhan result is proved also in a functional space smaller than $M_b(\mathcal{S})$. This is the content of the next proposition. We refer to **section C1** in *Dupuis and Ellis (1997)*.

Proposition D.4.1. *Given $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{S})$, we have the following equality.*

$$\sup_{g\in M_b(\mathcal{S})} \Big\{ \int_{\mathcal{S}} g(x)\mathbb{P}(dx) - \ln \int_{\mathcal{S}} e^{g(x)}\mathbb{Q}(dx) \Big\} = \sup_{g\in C_b(\mathcal{S})} \Big\{ \int_{\mathcal{S}} g(x)\mathbb{P}(dx) - \ln \int_{\mathcal{S}} e^{g(x)}\mathbb{Q}(dx) \Big\}.$$

From **Theorem D.4.1** and **Theorem D.4.2** we concluded a relationship of convex duality between Laplace functionals and relative entropy. For the definition of convex conjugate functions we refer the reader to **Section A1** of Appendix.

Remark D.4.3. *Let $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$. Then*

$$R(\mathbb{P}||\mathbb{Q}) = \sup_{g\in M_b(\mathcal{S})} \Big\{ \int_{\mathcal{S}} g d\mathbb{P} - \ln \int_{\mathcal{S}} e^g d\mathbb{P} \Big\} \quad \text{for all } \mathbb{P} \in \mathcal{P}(\mathcal{S}),$$
$$\ln \int_{\mathcal{S}} e^g d\mathbb{Q} = \sup_{\mathbb{P}\in\mathcal{P}(\mathcal{S})} \Big\{ \int_{\mathcal{S}} d\mathbb{P} - R(\mathbb{P}||\mathbb{Q}) \Big\}, \quad \text{for all } g \in M_b(\mathcal{S}),$$

which means that the functions $\mathbb{P} \mapsto R(\mathbb{P}||\mathbb{Q})$ and $g \mapsto \ln \int_{\mathcal{S}} e^g d\mathbb{Q}$ are convex conjugates.

The following proposition states the interchange of the limit and the integral by requiring a weak convergence of probability measures and the uniform boundedness of their relative entropy with respect to a reference probability measure. We refer to **section C.1** of Appendix for the definition and properties of weak convergence of probability measures.

Proposition D.4.2 (**Limit theorem for integrals**). *Let $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ be a Polish Space with the associated Borel σ-algebra. Let $\mathbb{Q}$ be a probability measure defined in $\mathcal{S}$ and $f : \mathcal{S} \longrightarrow \mathbb{R}$ be a bounded measurable function.*
Consider a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{P}(\mathcal{S})$ such that

$$\sup_{n\in\mathbb{N}} R(\mathbb{P}_n||\mathbb{Q}) \le \alpha < +\infty.$$

Assume that $\mathbb{P}_n \Rightarrow \mathbb{P}$. Then the following hold:

i) $\lim_{n\to\infty} \int_{\mathcal{S}} f d\mathbb{P}_n = \int_{\mathcal{S}} f d\mathbb{P}.$

ii) If $(f_n)_{n\to\infty}$ is a family of bounded continuous convergent $\mathbb{Q}$ - a.s to f, then $\lim_{n\to\infty}\int_{\mathcal{S}} f_n d\mathbb{P}_n = \int_{\mathcal{S}} f d\mathbb{P}$.

Proof. 1. Let us first prove that $\mathbb{P} \ll \mathbb{Q}$.
Since $\mathbb{P}_n \Rightarrow \mathbb{P}$ and $R(.||\mathbb{Q})$ is a lower semicontinuous function (point 1. of **Lemma D.4.1**), we get that

$$R(\mathbb{P}||\mathbb{Q}) \leq \liminf_{n\to\infty} R(\mathbb{P}_n|\mathbb{Q}) \leq \alpha < \infty.$$

We conclude that $\mathbb{P} \ll \mathbb{Q}$. By definition of relative entropy we have that $\mathbb{P}_n \ll \mathbb{Q}$. In order to prove i) we recall a basic fact of Analysis. For a proof we refer the reader to *Doob (1994)*-**Theorem V.16a**.
Since f is a bounded measurable function there exists a sequence $(\tilde{f}_k)_{k\in\mathbb{N}}$ of bounded continuous functions such that

$$\lim_{k\in\mathbb{N}} \tilde{f}_k = f \quad \mathbb{Q}- \ a.s.$$

Since $\mathbb{P} \ll \mathbb{Q}$, the above convergence holds $\mathbb{P}$-a.s. too. We want want to prove that

$$\int_{\mathcal{S}} f d\mathbb{P}_n \to \int_{\mathcal{S}} f d\mathbb{P} \quad \text{as } k \to \infty.$$

Fixed $k \in \mathbb{N}$, since $\mathbb{P}_n \Rightarrow \mathbb{P}$, as $n \to \infty$, we have

$$\int_{\mathcal{S}} \tilde{f}_k d\mathbb{P}_n \to \int_{\mathcal{S}} \tilde{f}_k d\mathbb{P} \quad \text{as } n \to \infty.$$

By dominated convergence we conclude

$$\int_{\mathcal{S}} \tilde{f}_k d\mathbb{P} \to \int_{\mathcal{S}} f d\mathbb{P} \quad \text{as } k \to \infty.$$

In order to prove (i) it remains to show that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \int_{\mathcal{S}} |\tilde{f}_k - f| d\mathbb{P}_n \to 0.$$

Let us fix $\delta > 0$. Let $M > 0$ such that $||f||_\infty \leq M$ and $\sup_{k\in\mathbb{N}} ||f_k||_\infty \leq M$. Hence,

$$\begin{aligned}
\int_{\mathcal{S}} |\tilde{f}_k - f| d\mathbb{P}_n &= \int_{\{|\tilde{f}_k - f| > \delta\}} |\tilde{f}_k - f| d\mathbb{P}_n + \int_{\{|\tilde{f}_k - f| \leq \delta\}} |\tilde{f}_k - f| d\mathbb{P}_n \\
&\leq \int_{\{|\tilde{f}_k - f| > \delta\}} |\tilde{f}_k - f| d\mathbb{P}_n + \delta \\
&\leq 2M\mathbb{P}_n\{|\tilde{f}_k - f| > \delta\} + \delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary the statement follows if we prove

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}_n\{|\tilde{f}_k - f| > \delta\} = 0.$$

Fix $c \in (1, \infty)$. Then $\log c > 0$. It follows that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \mathbb{P}_n\{|\tilde{f}_k - f| > \delta\} \\
&= \sup_{n\in\mathbb{N}} \int_{|\tilde{f}_k - f|>\delta} \frac{d\mathbb{P}_n}{d\mathbb{Q}} d\mathbb{Q} \\
&\le \sup_{n\in\mathbb{N}} \int_{\{|\tilde{f}_k - f|>\delta\}\cap\{\frac{d\mathbb{P}_n}{d\mathbb{Q}}\le c\}} \frac{d\mathbb{P}_n}{d\mathbb{Q}} d\mathbb{Q} + \sup_{n\in\mathbb{N}} \int_{\{|\tilde{f}_k - f|>\delta\}\cap\{\frac{d\mathbb{P}_n}{d\mathbb{Q}}\ge c\}} \frac{d\mathbb{P}_n}{d\mathbb{Q}} d\mathbb{Q}.
\end{aligned}$$

The first integral in the last estimate vanishes as $k \to 0$, since the density $\frac{d\mathbb{P}_n}{d\mathbb{Q}}$ is bounded in the corresponding domain of integration and since $f_k \to f$ $\mathbb{Q}$-a.s. as $k \to \infty$.
About the second integral, writing

$$A := \Big\{|\tilde{f}_k - f| > \delta\Big\} \cap \Big\{\frac{d\mathbb{P}_n}{d\mathbb{Q}} \ge c\Big\},$$

we have

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \int_A \frac{d\mathbb{P}_n}{d\mathbb{Q}} d\mathbb{Q} &\le \sup_{n\in\mathbb{N}} \int_A \frac{d\mathbb{P}_n}{d\mathbb{Q}} \ln \frac{d\mathbb{P}_n}{d\mathbb{Q}} \frac{1}{\ln \frac{d\mathbb{P}_n}{d\mathbb{Q}}} d\mathbb{Q} \\
&\le \frac{1}{\ln c} \sup_{n\in\mathbb{N}} \int_A \frac{d\mathbb{P}_n}{d\mathbb{Q}} \ln \frac{d\mathbb{P}_n}{d\mathbb{Q}} d\mathbb{Q} \\
&\le \frac{1}{\ln c} \sup_{n\in\mathbb{N}} \int_{\mathcal{S}} \frac{d\mathbb{P}_n}{d\mathbb{Q}} \ln \frac{d\mathbb{P}_n}{d\mathbb{Q}} d\mathbb{Q} \\
&= \frac{1}{\ln c} R(\mathbb{P}_n||\mathbb{Q}) \\
&< \frac{\alpha}{\ln c} \to 0 \qquad \text{as } c \to \infty,
\end{aligned}$$

which finishes the proof of statement (i).

2. In order to obtain ii) we just observe that

$$\int_{\mathcal{S}} f_n d\mathbb{P}_n = \int_{\mathcal{S}} f d\mathbb{P}_n + \int_{\mathcal{S}} (f_n - f) d\mathbb{P}_n.$$

The first integral in the right-hand side converges to $\int_{\mathcal{S}} f d\mathbb{P}$ since $\mathbb{P}_n \Rightarrow \mathbb{P}$ as $n \to \infty$. The second term converges to zero following the arguments of the previous step 1, replacind f_k and f of the previous step by f_n and f and letting $n \to \infty$ instead of taking the supremum in $n \in \mathbb{N}$.

$\square$

D.5 A variational representation for functionals of Poisson random measures

Let $\mathcal{X}$ be a Polish space. If $(X^\varepsilon)_{\varepsilon>0}$ is a family of $\mathcal{X}$- valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ obeying the *Laplace-Varadhan principle* as stated in **Definition D.3.1** with good rate function I and speed $b(\varepsilon)$, by **Theorem D.3.2** $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle with good rate function I and speed $b(\varepsilon)$. In a nutshell, if $\mathbb{P}^\varepsilon := \mathbb{P} \circ (X^\varepsilon)^{-1}$ is the law of X^ε, for $\varepsilon > 0$, from **Theorem D.4.2** it follows, for all $h \in C_b(\mathcal{X})$

$$-b(\varepsilon) \ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}h(X^\varepsilon)}\Big] = \inf_{\mathbb{Q}\in\mathcal{P}(\mathcal{X})} \Big\{ \int_{\mathcal{S}} h(x)\mathbb{Q}(dx) + R(\mathbb{Q}||\mathbb{P}^\varepsilon) \Big\}.$$

The goal is to show, through some variational formulas, the convergence of the variational expression

$$\inf_{\mathbb{Q}\in\mathcal{P}(\mathcal{X})} \Big\{ \int_{\mathcal{S}} h(x)\mathbb{Q}(dx) + R(\mathbb{Q}||\mathbb{P}^\varepsilon) \Big\} \to \inf_{x\in\mathcal{X}} \Big\{ h(x) + I(x) \Big\} \qquad \text{as } \varepsilon \to 0.$$

This section follows closely *Budhiraja et al. (2011).*

D.5.1 Notation and controlled random measures

Let $\mathcal{X}$ be a locally compact Polish space. In all the work developed in this thesis we have $\mathcal{X} = \mathbb{R}^d$.
We denote by $\mathcal{M}(\mathcal{X})$ the space of the measures ν defined on $\mathcal{B}(\mathcal{X})$, the σ-algebra of the Borel sets of $\mathcal{X}$, such that

$$\nu(K) < \infty \qquad \text{for all compact sets } K \subset \mathcal{X}.$$

We endow $\mathcal{M}(\mathcal{X})$ with the weak convergence topology, that is the coarsest topology such that for all $f \in C_c(\mathcal{X})$, compactly supported continuous functions on $\mathcal{X}$, the functional

$$\mathcal{M}(\mathcal{X}) \ni \nu \mapsto \langle f, \nu \rangle := \int_{\mathcal{X}} f(x)\nu(\mathrm{d}x) \in \mathbb{R}$$

is continuous. This topology can be metrized such that $\mathcal{M}(\mathcal{X})$ is a Polish space. We refer to **Subsection D.5.3** for the description of the topology. We fix $T > 0$ ($T = \infty$ eventually),

$$\mathcal{X}_T := [0, T] \times \mathcal{X},$$

and a non-atomic measure $\nu \in \mathcal{M}(\mathcal{X})$. Let ds be the Lebesgue measure on $[0, T]$. We define the measure

$$\nu_T := ds \otimes \nu.$$

Notation: From now on we fix $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space $(\mathcal{M}(\mathcal{X}_T), \mathcal{B}(\mathcal{M}(\mathcal{X}_T)), \mathbb{P})$ such that the canonical map

$$N : \mathcal{M}(\mathcal{X}_T) \longrightarrow \mathcal{M}(\mathcal{X}_T)$$

$$N(m) := m$$

is a Poisson random measure with intensity measure ν_T. Given $\theta > 0$ we consider the canonical probability measure $\mathbb{P}^\theta$ on $(\mathcal{M}(\mathcal{X}_T), \mathcal{B}(\mathcal{M}(\mathcal{X}_T)))$ under which N is a Poisson random measure with intensity measure $\theta\nu_T$. The corresponding expectation will be denoted by $\mathbb{E}^\theta$.
We augment the space of increments $\mathcal{X}$ by jump intensities with values on $[0, \infty)$ and define

$$\mathcal{Y} = \mathcal{X} \times [0, \infty).$$

Analogously we define

$$\mathcal{Y}_T = [0, T] \times \mathcal{Y}.$$

We consider instead of the Poisson random measure N with intensity ν_T, the Poisson random measure $\bar{N}$, whose intensity is $\nu_T \otimes dr$, where dr denotes the Lebesgue measure on $[0, \infty)$. The desired jump intensities can then be obtained by thinning this variable. Let $\mathcal{M}(\mathcal{Y}_T)$ be the space of locally finite measures defined on the augmented space $\mathcal{Y}_T$ and let $\bar{\mathbb{P}}$ be the unique probability measure on $(\mathcal{M}(\mathcal{Y}_T), \mathcal{B}(\mathcal{M}(\mathcal{Y}_T)))$ under which the canonical map

$$\bar{N} : \mathcal{M}(\mathcal{Y}_T) \longrightarrow \mathcal{M}(\mathcal{Y}_T)$$

$$\bar{N}(\bar{m}) := \bar{m}$$

is a Poisson random measure with intensity

$$\bar{\nu}_T := ds \otimes \nu \otimes dr,$$

with dr denoting the Lebesgue measure on $[0, \infty)$.The corresponding expectation will be denoted by $\bar{\mathbb{E}}$. We define the σ-algebra generated by $\bar{N}$,

$$\mathcal{F}_t := \sigma\{\bar{N}((0, s] \times A) : 0 \leq s \leq t, A \in \mathcal{B}(\mathcal{Y})\}$$

and let $(\bar{\mathcal{F}}_t)_{t \geq 0}$ denote the completion of $(\mathcal{F}_t)_{t \geq 0}$ under $\bar{\mathbb{P}}$.
We denote by $\bar{\mathcal{P}}$ the predictable σ-field on $[0, T] \times \mathcal{M}(\mathcal{Y}_T)$ with filtration $(\bar{\mathcal{F}}_t)_{0 \leq t \leq T}$, i.e. the σ-algebra generated on $[0, T] \times \mathcal{M}(\mathcal{Y}_T)$ by all $\{\bar{\mathcal{F}}_t\}$-adapted càdlàg processes.
We define the following functional space

$$\bar{\mathcal{A}}^+ := \Big\{\varphi : \mathcal{X}_T \times \mathcal{M}(\mathcal{Y}_T) \longrightarrow [0, \infty) \mid \varphi \text{ is } (\bar{\mathcal{P}} \otimes \mathcal{B}(\mathcal{X}), \mathcal{B}([0, +\infty))) \text{ measurable}\Big\}.$$

For a given function $\varphi \in \bar{\mathcal{A}}^+$, we define in this context the controlled random measure

$$N^\varphi((0, t] \times U) = \int_0^t \int_U \int_0^\infty 1_{[0, \varphi(s, z, \bar{m})]}(r) \bar{N}(\mathrm{d}s\mathrm{d}z\mathrm{d}r) \quad \text{for all } t \in [0, T],\, U \in \mathcal{B}(\mathcal{X}). \tag{D.5.1}$$

We call N^φ a controlled random measure by the random control $\varphi \in \bar{\mathcal{A}}^+$.

Remark D.5.1.

(i) N^{φ} is a controlled random measure, with φ selecting in a random but adapted way the intensity for the points at increment x and time s. This construction is made to control how to drive the intensity at time t, corresponding to the jump increment z, from the underlying Poisson random measure $\bar{N}$ to essentially any value on $[0, \infty)$.

(ii) Given $\varepsilon > 0$, we define the compensated Poisson random measure with intensity measure $\frac{1}{\varepsilon} \otimes \nu$ by

$$\tilde{N}^{\varepsilon^{-1}}((0,t] \times B) = N^{\varepsilon^{-1}}((0,t] \times B) - \frac{1}{\varepsilon} t\nu(B)$$

for all $B \in \mathcal{B}(\mathcal{X})$.

iii) Let us define $h : \mathcal{M}(\mathcal{Y}_T) \longrightarrow \mathcal{M}(\mathcal{X}_T)$ with

$$h(\bar{m})(U \times (0,T]) := \int_{U\times(0,t]\times(0,\infty)} \mathbf{1}_{[0,1]}(r)\bar{m}(ds, dz, dr), \text{ for all } t \in [0,T], \quad U \in \mathcal{B}(\mathcal{X}). \tag{D.5.2}$$

Hence, with the constant control $\varphi \equiv 1$, the corresponding controlled random measure N^1 is given as a functional of the underlying Poisson random measure $\bar{N}$ as

$$N^1 = h(\bar{N}).$$

We define the real valued entropy function

$$\begin{aligned} \ell &: [0,\infty) \longrightarrow [0,\infty) \\ \ell(r) &= r \log r - r + 1 \end{aligned} \tag{D.5.3}$$

and the entropy functional,

$$\mathfrak{L}_T : \bar{\mathcal{A}}^+ \longrightarrow [0, \infty]$$

$$\mathfrak{L}_T(\varphi) := \int_0^T \int_{\mathcal{X}} (\varphi(s,z) \ln \varphi(s,z) - \varphi(s,z) + 1)\nu(dz)ds, \tag{D.5.4}$$

for any $\varphi \in \bar{\mathcal{A}}^+$.

D.5.2 Auxiliary results and the variational principle for Laplace functionals of Poisson random mesures

We state the variational formula for Laplace functionals of Poisson Random measures.

Theorem D.5.1. *For all $F \in M_b(\mathcal{X}_T)$, bounded measurable function in $\mathcal{X}_T$, and for every $\theta > 0$, we have*

$$-\ln \mathbb{E}^{\theta}[e^{-F(N)}] = -\ln \bar{\mathbb{E}}[e^{-F(N^{\theta})}] = \inf_{\varphi \in \bar{\mathcal{A}}^{+}} \{\bar{\mathbb{E}}[\theta \mathfrak{L}_T(\varphi)] + F(N^{\theta\varphi})]\}.$$

Remark D.5.2. *From **Remark D.5.1**, the first equality stated in the theorem is clear.*

In order to prove **Theorem D.5.1** we introduce the following subset of $\bar{\mathcal{A}}^{+}$, $\bar{\mathcal{A}}^{+}_{b}$ that we call class of **nice controls**. We proceed as follows.

- Consider the exhaustion $(K_n)_{n\in\mathbb{N}}$ of compact sets $K_n \subset \mathcal{X}$ with

$$\bigcup_{n\in\mathbb{N}} K_n = \mathcal{X}.$$

- For $n \in \mathbb{N}$ define

$$\begin{aligned}\bar{\mathcal{A}}^{+}_{b,n} := \{\varphi \in \bar{\mathcal{A}}^{+} \mid \ &\text{for all } (t,\bar{m}) \in [0,T] \times \bar{\mathcal{M}}(\mathcal{Y}_T) : \\ &\frac{1}{n} \le \varphi(t,x,\bar{m}) \le n, \text{ if } x \in K_n, \quad \varphi(t,x,\bar{m}) = 1 \text{ if } x \in K_n^c\}.\end{aligned}$$

- Let

$$\bar{\mathcal{A}}^{+}_{b} := \bigcup_{n\in\mathbb{N}} \bar{\mathcal{A}}^{+}_{b,n}.$$

  Define $\tilde{N}(A) := N^1(A) - \nu_T(A)$, $A \in \mathcal{B}(\mathcal{X}_T)$ such that $\nu_T(A) < \infty$.
  Define $\bar{\tilde{N}}(A) := \bar{N}^1(G) - \bar{\nu}_T(G)$, $G \in \mathcal{B}(\mathcal{Y}_T)$ such that $\bar{\nu}_T(G) < \infty$.

- Similarly, let

$$\begin{aligned}\hat{\mathcal{A}}^{+}_{b} := \{\vartheta : \mathcal{Y}_T \times \bar{\mathcal{M}}(\mathcal{Y}_T) \longrightarrow \mathbb{R}^{+} \mid \vartheta \text{ is } (\mathcal{P} \otimes \mathcal{B}(\mathcal{Y}), \mathcal{B}(\mathbb{R}^{+})) \text{ measurable, with} \\ \vartheta(t,x,r,\bar{m}) = 0, \text{ whenever } (x,r) \in K^c \text{ for some compact set } K\}.\end{aligned}$$

  For every $\varphi \in \bar{\mathcal{A}}^{+}_{b}$ we define the *Doleans-Dade exponential* of φ,

$$\begin{aligned}&\mathcal{E}_t(\varphi)(\bar{m}) \\ &:= \exp\Big(\int_{(0,t]\times\mathcal{X}} \ln(\varphi(s,x,\bar{m}))N^1_c(ds,dx)(\bar{m}) \\ &+ \int_{(0,t]\times\mathcal{X}} (\ln(\varphi(s,x,\bar{m})) - \varphi(s,x) + 1)\nu_T(ds,dx)\Big) \\ &= \exp\Big(\int_{(0,t]\times\mathcal{X}\times[0,1]} \ln(\varphi(s,x,\bar{m}))N(ds,dx)(\bar{m}) \\ &+ \int_{(0,t]\times\mathcal{X}\times[0,1]} (-\varphi(s,x) + 1)\bar{\nu}_T(ds,dx)\Big).\end{aligned} \tag{D.5.5}$$

The following lemma is direct consequence from the *Girsanov theorem* stated in **Theorem B.3.2** in **Appendix B** and it is used in the proof of **Theorem D.5.1**.

Lemma D.5.1 (**Change of measure for** $\bar{N}$)**.** *Let* $\varphi \in \bar{\mathcal{A}}_b^+$*. Then*

- $(\mathcal{E}_t(\varphi))_{t\geq 0}$ *is a* $(\bar{\mathcal{F}}_t)_{t\in[0,T]}$ *martingale with respect to* $\bar{\mathbb{P}}$*.*
- *Define* $\mathbb{Q}^\varphi$ *on* $\mathcal{M}(\mathcal{Y}_T)$ *by*

$$\mathbb{Q}^\varphi(G) := \int_G \mathcal{E}_T(\varphi) d\bar{\mathbb{P}}, \qquad \textit{for } G \in \mathcal{B}(\mathcal{M}(\mathcal{Y}_T)).$$

- *For any* $\vartheta \in \hat{\mathcal{A}}_b$

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^\varphi}\left[\int_{\mathcal{Y}_T} \vartheta(s,x,r)\bar{N}(ds,dz,dr)\right]\\ &= \mathbb{E}^{\mathbb{Q}^\varphi}\left[\int_{\mathcal{Y}_T} \vartheta(s,x,r)[\varphi(s,x)\mathbf{1}_{(0,1]}(r) + \mathbf{1}_{(1,\infty)}(r)]\bar{\nu}_T(ds,dz,dr)\right].\end{aligned}$$

Remark D.5.3. *The last statement says that under* $\mathbb{Q}^\varphi$*,* $\bar{N}$ *is a random counting measure with compensator* $[\varphi(s,z)\mathbf{1}_{(0,1]}(r) + \mathbf{1}_{(1,\infty)(r)}]\bar{\nu}_T(ds,dz,dr)$*.*

The next lemma is a result of approximation for the class of controls that we introduced before and it isused to prove the upper bound of the variational formula of the **Theorem D.5.1**. We refer the reader to *Budhiraja et al. (2011)* **-Lemma 2.4** for the proof. The proof uses standard measure-theoretical arguments and we present only a sketch.

Lemma D.5.2 (**Approximation of the nice controls**)**.** *Let* $n \in \mathbb{N}$ *and* $\varphi \in \bar{\mathcal{A}}_{b,n}^+$*. Then there exists a sequence of processes* $\varphi_k \in \bar{\mathcal{A}}_{b,n}^+$ *with the following properties.*

1. *Fix* $n \in \mathbb{N}$*. Then for each* $k \in \mathbb{N}$*, there exist* $n_1, \ldots, n_k \in \mathbb{N}$*,* $0 = t_0 < t_1 < \cdots < t_k = T$ *and a family of non-negative random variables* $(X_{ij})_{i=1,\ldots,k;j=1,\ldots,n_k}$ *such that* X_{ij} *is* $\bar{\mathcal{F}}_{t_i}$*-measurable satisfying* $\frac{1}{n} \leq X_{ij} \leq n$ *and a measurable disjoint partition* $(E_{ij})_{\substack{i=1,\ldots,k\\ j=1,\ldots,n_k}}$ *of* K_n *fullfilling*

$$\varphi_k(t,z,\bar{m}) = \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{k}\sum_{j=1}^{n_\ell} \mathbf{1}_{(t_{i-1},t_i]}(t) X_{ij}(\bar{m})\mathbf{1}_{E_{ij}}(z) + \mathbf{1}_{K_n^c}(z)\mathbf{1}_{(0,T]}(t).$$

2. N^{φ_k} *converges in distribution to* N^φ *as* $k \to \infty$*.*
3. $\bar{\mathbb{E}}|\mathfrak{L}_T(\varphi_k) - \mathfrak{L}_T(\varphi)| \to 0$ *and* $\bar{\mathbb{E}}|\mathcal{E}_T(\varphi_k) - \mathcal{E}_T(\varphi)| \to 0$*, as* $k \to \infty$*.*

Sketch of the proof.

i) We start to define such sequence and prove statement 2. of the lemma. For each $k \in \mathbb{N}$, let us define

$$\varphi_k(t,z,\bar{m}) := \frac{k}{n}\Big(\frac{1}{k} - t\Big)^+ + k\int_{(t-\frac{1}{k})^+}^{t} \varphi(s,z,\bar{m})ds \quad \text{for } (t,z,\bar{m}) \in \mathcal{X}_T \times \mathcal{M}(\mathcal{Y}_T). \tag{D.5.6}$$

Using *Lusin's theorem* (*Rudin (1966)*- pp.55) we have that $(z,\bar{m})$ $\nu \otimes \bar{\mathbb{P}}$ - a.e., , as $k \to \infty$

$$\begin{aligned} &\int_0^T |\varphi_k(t,z,\bar{m}) - \varphi(t,z,\bar{m})|dt \to 0 \\ &\int_0^T |\ell(\varphi_k(t,z,\bar{m})) - \ell(\varphi(t,z,\bar{m}))|dt \to 0. \end{aligned} \tag{D.5.7}$$

For every $k \in \mathbb{N}$, $\varphi_k \in \bar{\mathcal{A}}^+_{b,n}$ and

$$\bar{\mathbb{E}}[|\mathfrak{L}_T(\varphi_k) - \mathfrak{L}_T(\varphi)|] \to 0 \quad \text{as } k \to \infty.$$

Given $f \in C_c(\mathcal{X}_T)$, for some K_n of the exhaustion of $\mathcal{X}$ considered before,

$$\begin{aligned} \bar{\mathbb{E}}[|\langle f, N^{\varphi_k}\rangle| - \langle f, N^{\varphi}\rangle|] &\le \bar{\mathbb{E}}\Big[|f(s,z)||\mathbf{1}_{[0,\varphi_k(s,z,\bar{m})]}(r) - \mathbf{1}_{[0,\varphi(s,z,\bar{m})]}(r)|\bar{\nu}_T(ds,dz,dr)\Big] \\ &\le ||f||_\infty \bar{\mathbb{E}}\Big[\int_{[0,T]\times K_n} |\varphi_k(t,z,\bar{m}) - \varphi(t,z,\bar{m})|\nu_T(ds,dz).\Big]. \end{aligned}$$

Thanks to (D.5.7) and due to $\nu(K_n) < \infty$ we conclude that the last right-hand side of the previous estimates converges to 0, as $k \to \infty$. Therefore $N^{\varphi_k} \Rightarrow N^{\varphi}$.

i) We check the statement 3. of the lemma in the next lines. To show the convergence in $L^1(\bar{\mathbb{P}})$ of the *Doleans-Dade exponentials* $(\mathcal{E}_T(\varphi_k))_{k\in\mathbb{N}}$ to $\mathcal{E}_T(\varphi)$, by *Scheffe's lemma* (**Proposition C.1.3**) it is enough to check that

$$\mathcal{E}_T(\varphi_k) \to \mathcal{E}_T(\varphi) \quad \bar{\mathbb{P}} - \text{a.s. as } k \to \infty.$$

By definition of *Doleans-Dade* (D.5.5) it is enough to check the convergence $\bar{\mathbb{P}}$-a.s. of

$$\begin{aligned} &\int_0^T\int_{\mathcal{X}} (1 - \varphi_k(s,z))\nu_T(ds,dz) \to \int_0^T\int_{\mathcal{X}} (1 - \varphi(s,z))\nu_T(ds,dz) \quad \text{and} \\ &\int_0^T\int_{\mathcal{X}} \ln(\varphi_k(s,x))N^1(ds,dz) \to \int_0^T\int_{\mathcal{X}} \ln(\varphi(s,z))N^1(ds,dz), \quad \text{as } k \to \infty. \end{aligned}$$

The first convergence is immediate from (D.5.7), combined with the uniform bounds on φ, φ_k, $\nu(K_n) < \infty$ and the fact that $1 - \varphi_k = 1 - \varphi = 0$ on K_n^c.

The second convergence follows from the previous considerations and from the elementar estimate

$$|\ln(\varphi_k(s,z)) - \ln(\varphi(s,z))| \leq n|\varphi_k(s,z) - \varphi(s,z)|.$$

The above considerations prove

$$\mathcal{E}_T(\varphi_k) \to \mathcal{E}_T(\varphi) \quad \bar{\mathbb{P}} - \text{a.s.} \quad k \to \infty.$$

By *Scheffe's lemma* the convergence in $L^1(\bar{\mathbb{P}})$ of $(\mathcal{E}_T(\varphi_k))_{k\in\mathbb{N}}$ to $\mathcal{E}_T(\varphi)$ is assured.

iii) We prove the first statement of the lemma.
By construction

$$t \mapsto \varphi_k(t,z,\bar{m}) \text{ is continuous } \nu \otimes \bar{\mathbb{P}}\text{-a.e.}$$

Fixed $k \in \mathbb{N}$ and $q \in \mathbb{N}$, we define

$$\varphi_k^q(t,z,\bar{m}) := \sum_{m=0}^{\lfloor qT \rfloor} \varphi_k\Big(\frac{m}{q}, z, \bar{m}\Big)\mathbf{1}_{(\frac{m}{q}, \frac{m+1}{q}]}(t), \quad (t,z,\bar{m}) \in \mathcal{X}_T \times \mathcal{M}(\mathcal{Y}_T).$$

It can be checked that $(\varphi_k^q)_{q\in\mathbb{N}}$ satisfies statements 2. and 3 of the lemma.
Fixed $q, m \in \mathbb{N}$,

$$g(z,\bar{m}) := \varphi_k\Big(\frac{m}{q}, z, \bar{m}\Big) \text{ is } \mathcal{B}(\mathcal{X}) \otimes \bar{\mathcal{F}}_{\frac{m}{q}} - \text{measurable}$$

with

$$g(z,\bar{m}) \in \Big[\frac{1}{n}, n\Big] \text{ and } g(z,\bar{m}) = 1 \text{ when } z \in K_n^c.$$

Using measure-theoretic arguments it can be built a $\mathcal{B}(\mathcal{X}) \otimes \bar{\mathcal{F}}_{\frac{m}{q}}$-measurable sequence of maps $(g_r)_{r\in\mathbb{N}}$ with the following properties:

- the sequence satisfies $g_r(z,\bar{m}) = \sum_{j=1}^{a(r)} c_j^r(\bar{m})\mathbf{1}_{E_j^r}(z)$ for $z \in K_n$
  where $(E_j^r)_{j=1}^{a(r)}$ is some measurable partition of K_n;
- the coefficients $c_j^r(\bar{m}) \in \Big[\frac{1}{n}, n\Big]$ $\bar{\mathbb{P}}$ -a.s.;
- the functions $g_r(z,\bar{m}) = 1$ for all $z \in K_n^c$ and
- the convergence $g_r \to g$ as $r \to \infty$ is assured $\nu \otimes \bar{\mathbb{P}}$- a.e
  Taking q, r sufficiently large, we can produce from this third level of approximation other approximation of φ that satisfy the statements of the lemma.

$\square$

Denote the set of processes with this representation as $\bar{\mathcal{A}}^+_{s,n}$.
Let

$$\bar{\mathcal{A}}^+_s := \bigcup_{n=1}^{\infty} \bar{\mathcal{A}}^+_{s,n}.$$

We wcall this class the class of **simple processes**.

The following lemma is crucial in the proof of **Theorem D.5.1**. It permits to evaluate the expected values of the desired functionals of the controlled Poisson random measures, changing the probability space by means of changing the measure by the one which density is the *Doleans-Dade exponential* $\mathcal{E}$. We do not present a proof of this result. The proof relies on techniques of approximation of the class of simple controls introduced in **Lemma D.5.2** and on the decomposition of the underlying random measures in small time intervals where the statements of lemma are verified, doing a passage to the limit afterwards. We refer the reader to *Budhiraja et al. (2011)*- **Lemma 2.5** (2011).

Lemma D.5.3 (Duality). *For any $\varphi \in \bar{\mathcal{A}}^+_s$, there is $\bar{\varphi} \in \bar{\mathcal{A}}^+_s$ such that*

$$\bar{\mathbb{P}} \circ (N^{\varphi})^{-1} = \mathbb{Q}^{\tilde{\varphi}} \circ (N^1)^{-1},$$

and

$$\mathbb{E}^{\bar{\mathbb{Q}}^{\bar{\varphi}}}\Big[\mathcal{L}_T(\bar{\varphi}) + F(N^1)\Big] = \bar{\mathbb{E}}\Big[\mathcal{L}_T(\varphi) + F(N^{\varphi})\Big]. \tag{D.5.8}$$

Conversely, given $\tilde{\varphi} \in \bar{\mathcal{A}}^+_s$ there is $\varphi \in \bar{\mathcal{A}}^+_s$ such that

$$\bar{\mathbb{P}} \circ (N^{\varphi})^{-1} = \bar{\mathbb{Q}}^{\tilde{\varphi}} \circ (N^1)^{-1}.$$

We introduce now the space of cylindric functions on $(\Omega, \mathcal{F}, \mathbb{P})$. Our probability space is $(\mathcal{M}(\mathcal{X}_T), \mathcal{B}(\mathcal{M}(\mathcal{X}_T)))$ as it was stated in the last section. We define

$$\begin{aligned} C_{cyl}(\mathcal{M}(\mathcal{X}_T)) := \Big\{ F : \mathcal{M}(\mathcal{X}_T) \longrightarrow \mathbb{R} \mid F(m) = h(\langle f_1, m\rangle, \dots \langle f_k, m\rangle), \\ k \in \mathbb{N}, h \in C_c^{\infty}(\mathbb{R}^k), f_i \in C_c(\mathcal{X}_T). \Big\}, \end{aligned}$$

where the following notation stands for the usual dual pairing,

$$\langle f, m\rangle = \int_{\mathcal{X}_T} f dm, \qquad \text{for all } f \in C_c(\mathcal{X}_T), m \in \mathcal{M}(\mathcal{X}_T).$$

It is a standard fact that $C_{cyl}(\mathcal{M}(\mathcal{X}_T))$ is dense in $\mathbb{M}_b(\mathcal{M}(\mathcal{X}_T))$ with respect to the topology of pointwise convergence. This is the content of the next lemma and it is used to prove the lower bound of the variational principle via an approximation argument. Since this is a standard fact we sketch only a proof.

Lemma D.5.4. *Let $F \in M_b(\mathcal{M}(\mathcal{X}_T))$ be a bounded random variable defined on the probability space $(\mathcal{M}(\mathcal{X}_T), \mathcal{B}(\mathcal{M}(\mathcal{X}_T)), \mathbb{P})$. Then there exists a family of functions $(F_n)_{n\in\mathbb{N}} \subset C_{cyl}(\mathcal{M}(\mathcal{X}_T))$ with*

$$\sup_{n\in\mathbb{N}} ||F_n||_\infty \leq ||F||_\infty,$$

such that

$$F_n(m) \to F(m) \quad \mathbb{P} - a.s. \quad n \to \infty.$$

Proof. We denote $C_0(\mathcal{X}_T)$ the completion of $C_c(\mathcal{X}_T)$ with respect to the uniform topology, which turns to be a separable Banach space. Let $(f_k)_{k\in\mathbb{N}}$ be a countable dense subset of $C_0(\mathcal{X}_T)$. We define the countable filtration of σ-algebras, for every $n \in \mathbb{N}$,

$$\mathcal{G}_n := \sigma\{\langle f_k, m\rangle \mid k = 1, \dots, n\}.$$

If follows that $\mathcal{G}_n \nearrow \mathcal{F} = \mathcal{B}(\mathcal{M}(\mathcal{X}_T))$. Let $F \in \mathbb{M}_b(\mathcal{M}(\mathcal{X}_T))$. We proceed via an approximation argument. For $n \in \mathbb{N}$ let $F_n = \mathbb{E}[F|\mathcal{G}_n]$. Hence, there exists a bounded measurable function h_n defined on $\mathbb{R}^d$ such that

$$G_n(m) = h_n(\langle f_1, m\rangle, \dots \langle f_n, m\rangle).$$

The next step is to use standard localizing and mollifying techniques, approximating h_n by $(h_{n_k})_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$. Using a diagonalization argument and up to a subsequence, we get the desired result. □

D.5.3 The compactness of the space S^M

For $M \geq 0$ let

$$S^M := \{g : \mathcal{X}_T \to [0, \infty) \mid \mathfrak{L}_T(g) \leq M\}.$$

A function $g \in S^M$ can be identified, as a density w.r.t. to ν_T, with $\nu_T^g \in \mathcal{M}(\mathcal{X}_T)$, given by

$$\nu_T^g(A) := \int_A g(s, z)\nu_T(ds, dz), \qquad \text{for all } A \in \mathcal{B}(\mathcal{X}_T).$$

We argue next that this identification turns S^M into a compact space.

Proposition D.5.1. *The identification stated above induces a topology on S^M under which S^M is a compact space.*

Proof. We argue that, for any $M \geq 0$, $S^M \simeq \{\nu_T^g : g \in S^M\}$ is a compact space. We sketch a proof of this statement in order to make the text self-contained. This fact is proved in *Budhiraja et al. (2013)* .
We first give a glimpse in how the topology on $\mathcal{M}(\mathcal{X}_T)$ can be metrized. Consider a sequence of open sets $\{O_j\}_{j\in\mathbb{N}}$ of $\mathcal{X}_T$ such that

i) $\bar{O}_j \subset O_{j+1}$,

ii) $\bar{O}_j$ is compact ,

ii) $\bigcup_{j\geq 1} O_j = [0,T]\times \mathcal{X}$

We consider $\varphi_j(x) = (1 - d(x, O_j)) \vee 0$ where d is the distance on $\mathcal{X}_T$.
Given $\mu \in \mathcal{M}(\mathcal{X}_T)$, we define $\mu^j \in \mathcal{M}(\mathcal{X}_T)$ by $\frac{d\mu^j}{d\mu}(x) = \varphi_j(x)$.
Given $\mu, \nu \in \mathcal{M}(\mathcal{X}_T)$ let

$$\bar{d}(\mu,\nu) := \sum_{j=1}^{+\infty} \frac{1}{2^j} ||\mu^j - \nu^j||_{BL},$$

where the norm considered is

$$||\mu^j - \nu^j||_{BL}$$
$$:= \sup\Big\{ \int_{\mathcal{X}_T} f d\mu^j - \int_{\mathcal{X}_T} f d\nu^j \mid ||f||_\infty \leq 1, |f(x) - f(y)| \leq d(x,y) \text{ for all } x, y \in \mathcal{X}_T \Big\}.$$

It can be shown that $\bar{d}$ defines a metric under which $\mathcal{M}(\mathcal{X}_T)$ is a Polish space. Convergence in this metric is essentially equivalent to weak convergence in each compact set of $\mathcal{M}(\mathcal{X}_T)$, which is what the following equivalence means:

$$\bar{d}(\mu_n,\mu) \to 0 \Leftrightarrow \forall f \in C_b(\mathcal{X}_T) \int_{\mathcal{X}_T} f d\mu_n^j \to \int_{\mathcal{X}_T} f d\mu^j, \quad n \to \infty.$$

We show finally that $\{\mu_T^g : g \in S^M\}$ is compact in $\mathcal{M}(\mathcal{X}_T)$. For every $n \in \mathbb{N}$, let $\mu^n = \nu_T^{g_n}$, for some $g_n \in S^M$.

1. We show that $\{\mu_n\} \subset \mathcal{M}(\mathcal{X}_T)$ is relatively compact for any sequence (g_n) in S^M. For this, using a diagonalization method, it is sufficient to prove that $(\mu_n^j) \subset \mathcal{M}(\mathcal{X}_T)$ is relatively compact for all $j \in \mathbb{N}$. The measure μ_n^j is supported compactly on the compact set of $\mathcal{X}$ given by $K^j = \overline{\{x : \varphi_j(x) \neq 0\}}$. In order to prove that $\{\mu_n^j\} \subset \mathcal{M}(\mathcal{X}_T)$ is relatively compact it is sufficient to prove that

$$\sup_{n\in\mathbb{N}} \mu_n^j(\mathcal{X}_T) < \infty.$$

Let us fix $C \in (0, +\infty)$ such that $z \leq C(\ell(z) + 1)$ for all $z \in (0, +\infty)$. The existence of such $C > 0$ is guaranteed by the superlinearity of the function ℓ (**Lemma D.7.1**). Observing that $\mathfrak{L}_T(g_n) \leq M$ we get that

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \mu_n^j(\mathcal{X}_T) &= \sup_{n\in\mathbb{N}} \int_{\mathcal{X}_T} \varphi_j(x) g_n(x) \nu_T(dx) \\
&\leq \sup_{n\in\mathbb{N}} \int_{K_j} \varphi_j(x) g_n(x) \nu_T(dx) \\
&\leq \sup_{n\in\mathbb{N}} \int_{K_j} C(\ell(g_n(x)) + 1) \nu_T(dx) \\
&\leq C \sup_{n\in\mathbb{N}} \mathfrak{L}_T(g_n) + C\nu_T(K_j) \\
&< \infty.
\end{aligned}$$

2. Suppose that along a subsequence $\mu_n \rightharpoonup \mu$ weakly. We intend to show that $\mu = \nu_T^g$, for some $g \in S^M$. If $\mu = 0$ there is nothing to prove. We now suppose that $\mu \neq 0$. Since $\mu_n \rightharpoonup \mu \neq 0$ as $n \to \infty$ we have that there exists $j_0 \in \mathbb{N}$ such that $\inf_{n\in\mathbb{N}} \nu_T^{g_n}(\bar{O}_j) > 0$ for all $j \geq j_0$. For all $j \geq j_0$, we define

$$\begin{aligned} c^j &:= \nu_T^j(\mathcal{X}_T) \quad \text{and } \bar{\nu}_T^j = \frac{\nu_T^j}{c^j}; \\ c_n^j &:= \mu_n^j(\mathcal{X}_T) \quad \text{and } \bar{\mu}_n^j := \frac{\mu_n^j}{c_n^j}; \\ c_\mu^j &:= \mu^j(\mathcal{X}_T) \quad \text{and } \bar{\mu}^j := \frac{\mu^j}{c_\mu^j}. \end{aligned}$$

The measures $\bar{\nu}_T^j, \bar{\mu_n}^j, \bar{\mu}^j$ constructed above are probability measures. Calculating the relative entropy between $\bar{\mu}_n^j$ and $\bar{\nu_T}^j$, due to the definition of ℓ, it follows

$$\begin{aligned} R(\bar{\mu}_n^j||\bar{\nu_T}^j) &= \frac{1}{c_n^j}\int_{\mathcal{X}_T} (\ln g_n(x) + \ln \frac{c^j}{c_n^j}) g_n(x)\varphi_j(x)\nu_T(dx) \\ &= \frac{1}{c_n^j}\int_{\mathcal{X}_T} [\ell(g_n(x)) + g_n(x) - 1]\varphi_j(x)\nu_T(dx) + \ln \frac{c^j}{c_n^j} \\ &\leq \frac{1}{c_n^j} M + 1 - \frac{c^j}{c_j^n} + \ln \frac{c^j}{c_j^n} \quad \text{as } n \to \infty. \end{aligned}$$

The convergence $\mu_n^j \Rightarrow \mu^j$ yields $c_n^j \to c_\mu^j$. Lower semicontinuity of the relative entropy (**Lemma D.4.1**) implies, for all $j \geq j_0$,

$$\begin{aligned} R(\bar{\mu}^j||\bar{\nu}_T^j) &\leq \liminf_{n\to+\infty} R(\bar{\mu}_n^j||\bar{\nu}_T^j) \\ &\leq \liminf_{n\to\infty} (\frac{1}{c_n^j} M + 1 - \frac{c^j}{c_n^j} + \ln \frac{c^j}{c_n^j}) \\ &\leq \frac{1}{c_\mu^j} M + 1 - \frac{c^j}{c_\mu^j} + \ln \frac{c^j}{c_\mu^j} \\ &< \infty. \end{aligned}$$

Therefore, we have $\mu^j \ll \nu_T^j$ for all $j \geq j_0$. We define $g^j(x) = \dfrac{d\mu^j}{d\nu_T^j}(x)$ and $g = g^j$ on $\bar{O}_j$. Due to the properties of $(O_j)_{j\in\mathbb{N}}$, g is well defined. It follows that$\mu = \nu_T^g$ and $g \in S^M$.

□

D.5.4 Proof of the variational principle

For the sake of readibility we present the proof of **Theorem D.5.1** for the case $\theta = 1$. We start with the upper bound.

Theorem D.5.2 (Upper bound of Theorem D.5.1). *For any* $F \in M_b(\mathcal{M}(\mathcal{X}_T))$ *we have*

$$-\ln(\bar{\mathbb{E}}[e^{F(N^1)}]) \leq \inf_{\varphi \in \bar{\mathcal{A}}^+} \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi) + F(N^\varphi)].$$

Proof. It is our intent to prove the desired result using the variational representation of Laplace functionals in terms of relative entropies that is given in **Theorem D.4.1**. We start to evaluate $R(\mathbb{Q}^\varphi \mid\mid \bar{\mathbb{P}})$ for $\varphi \in \bar{\mathcal{A}}_b^+$.
From **Lemma D.5.1**, the *Doleans-Dade exponential* $(\mathcal{E}_t(\varphi))_{t\in[0,T]}$ is an $\{\bar{\mathcal{F}}_t\}_{t\in[0,T]}$ martingale with respect to $\bar{\mathbb{P}}$ and $\bar{N}$ is a counting random measure with compensator $[\varphi(s,x)\mathbf{1}_{(0,1]}(r) + \mathbf{1}_{(1,\infty)(r)}]\bar{\nu}_T(dsdxdr)$ under $\mathbb{Q}^\varphi$.

From the definition of relative entropy (**Definition D.4.1**) and due to the form of the relative entropy functional $\mathfrak{L}_T$ (**Definition D.5.4**) it follows that

$$\begin{aligned}
&R(\mathbb{Q}^\varphi||\bar{\mathbb{P}}) \\
&= \int_{\mathcal{M}(\mathcal{Y}_T)} \Big[\int_{\mathcal{X}_T} \ln(\varphi(s,x)(\bar{m}))N_c^1(ds,dx) \\
&+ \int_{\mathcal{X}_T} (\ln(\varphi(s,x)(\bar{m})) - \varphi(s,x)(\bar{m}) + 1)\nu_T(ds,dx)\Big]\mathbb{Q}^\varphi(d\bar{m}) \\
&= \int_{\mathcal{M}(\mathcal{Y}_T)} \Big[\int_{\mathcal{X}_T} \ln(\varphi(s,x)(\bar{m}))N^1(ds,dx) + \int_{\mathcal{X}_T} (-\varphi(s,x)(\bar{m}) + 1)\nu_T(ds,dx)\Big]\mathbb{Q}^\varphi(d\bar{m}) \\
&= \int_{\mathcal{M}(\mathcal{X}_T)} \Big[\int_{\mathcal{X}_T} (\varphi(s,x)(\bar{m})\ln(\varphi(s,x)(\bar{m})) - \varphi(s,x)(\bar{m}) + 1)\nu_T(ds,dx)\Big]\mathbb{Q}^\varphi(d\bar{m}) \\
&= \bar{\mathbb{E}}^{\mathbb{Q}^\varphi}[\mathfrak{L}_T(\varphi)].
\end{aligned} \tag{D.5.9}$$

Using **Theorem D.4.1** for $\mathbb{Q}^\varphi$ with $\varphi \in \bar{\mathcal{A}}_b^+$, (D.5.9) combined with the remark that N^1 can be written in terms of a functional of the underlying Poisson random measure as described in (D.5.2), we conclude that

$$\begin{aligned}
-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] &\leq \Big[R(\mathbb{Q}^\varphi||\bar{\mathbb{P}}) + \int_{\mathcal{M}(\mathcal{Y}_T)} F(h(\bar{m}))\mathbb{Q}^\varphi(d\bar{m})\Big] \\
&\leq \mathbb{E}^{\mathbb{Q}^\varphi}[\mathcal{L}_T(\varphi) + F(N^1)].
\end{aligned} \tag{D.5.10}$$

This shows that it suffices to prove that, for any $\varphi \in \bar{\mathcal{A}}^+$,

$$-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] \leq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi) + F(N^\varphi)]. \tag{D.5.11}$$

1. We start to prove (D.5.11) for simple bounded controls $\varphi \in \bar{\mathcal{A}}_s^+$, that are introduced respectively in **Lemma D.5.2**.
   Using the duality result given in **Lemma D.5.3** we can find $\bar{\varphi} \in \bar{\mathcal{A}}_s^+$, $(\bar{\mathcal{F}}_t)_{t\in[0,T]}$ predictable such that

   $$\mathbb{E}^{\mathbb{Q}^{\bar{\varphi}}}[\mathfrak{L}_T(\bar{\varphi}) + F(N^1)] = \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi) + F(N^\varphi)].$$

   Then (D.5.11) follows from (D.5.10).

2. We consider now the case when $\varphi \in \bar{\mathcal{A}}_b^+$ is a bounded control and we argue by approximation.
Given $\varphi \in \bar{\mathcal{A}}_b^+$, let $(\varphi_k)_{k\in\mathbb{N}} \in \bar{\mathcal{A}}_s^+$ be a sequence of simple controls that approximate φ as in the construction presented in **Lemma D.5.2**. Using the previous case 1., for all $k \in \mathbb{N}$, we have

$$-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] \leq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_k) + F(N^{\varphi_k})]. \tag{D.5.12}$$

Using statements 1. and 2. of **Lemma D.5.2**, under the probability $\bar{\mathbb{P}}$, $N^{\varphi_k} \Rightarrow N^{\varphi}$ and $\bar{\mathbb{E}}[\mathcal{L}_T(\varphi_k)] \to \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi)]$. But since F is not assumed to be continuous we cannot pass to the limit in the (D.5.12). We use instead the useful result of interchange of limit and integrals stated in **Proposition D.4.2**. The function F is bounded and measurable and there exists a uniform bound for the relative entropies $R(\bar{\mathbb{P}} \circ (N^{\varphi_k})^{-1}||\bar{\mathbb{P}}\circ(N^1)^{-1})$ and by **Proposition D.4.2** we can pass to the limit in (D.5.12). Let us evaluate the relative entropy laws N^{φ_k} with respect to $\bar{\mathbb{P}} \circ (N^1)^{-1}$,

$$\begin{aligned}
R(\bar{\mathbb{P}} \circ (N^{\varphi_k})^{-1}||\bar{\mathbb{P}} \circ (N^1)^{-1}) &= R(\mathbb{Q}^{\bar{\varphi}_k} \circ (N^1)^{-1}||\bar{\mathbb{P}} \circ (N^1)^{-1}) \\
&\leq R(\mathbb{Q}^{\bar{\varphi}_k}||\bar{\mathbb{P}}) \\
&= \mathbb{E}^{\mathbb{Q}^{\bar{\varphi}_k}}[\mathfrak{L}_T(\bar{\varphi}_k)] \\
&= \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_k)] \\
&\longrightarrow \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi)] \quad \text{for } k \to \infty \\
&< \infty.
\end{aligned} \tag{D.5.13}$$

In the computations above, the first equality follows from **Lemma D.5.3**. The subsequent inequality follows from the contraction property of relative entropy (**Lemma D.4.2**), which states that the relative entropy of a common push-forward (in this case the random variable N^1) of two probability measures never increases when the measures are induced by the same mapping. The second identity follows from the same argument used in (D.5.9) in the beginning of case 1. and the last one follows from **Lemma D.5.3**. The passage to the limit is justified by**Lemma D.5.2**. Then we conclude

$$\sup_k R(\bar{\mathbb{P}} \circ (N^{\varphi_k})^{-1}||\bar{\mathbb{P}} \circ (N^1)^{-1}) < \infty$$

and using the limit result of **Proposition D.4.1**, we can pass to the limit in (D.5.12) and conclude, that for all $\varphi \in \bar{\mathcal{A}}_b^+$,

$$-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] \leq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi) + F(N^{\varphi})]. \tag{D.5.14}$$

3. Given a general control $\varphi \in \bar{\mathcal{A}}^+$ we define the cut-off control

$$\varphi^n(x,t,\bar{m}) := \begin{cases} [\varphi(t,x,m) \vee 1/n] \wedge n & \text{if } x \in K_n, t \geq 0, \bar{m} \in \mathcal{M}(\mathcal{Y}_T), \\ 1 & \text{if else.} \end{cases}$$

We note that $\varphi^n \in \bar{\mathcal{A}}^+_{b,n}$, for all $n \in \mathbb{N}$ and that (D.5.12) holds true with φ_k replaced by φ^n. The definition of φ^n implies that $\varphi^n \geq 1$ and consequently $\ell(\varphi^n(t,x,\omega))$ is non decreasing. Using the monotone convergence theorem we conclude $\bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_n)] \nearrow \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi)] \quad n \to \infty$..

If $\bar{\mathbb{E}}[\mathfrak{L}_T(\varphi)] = \infty$ nothing has to be proved.
So let us assume now that $\bar{\mathbb{E}}[\mathfrak{L}_T(\varphi)] < \infty$.

Following the analogous computations as in (D.5.13), we obtain

$$R(\bar{\mathbb{P}} \circ (N^{\varphi^n})^{-1} || \bar{\mathbb{P}} \circ (N^1)^{-1}) \leq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi^n)] \leq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi)]. \tag{D.5.15}$$

The relative entropies in (D.5.15) are uniformly bounded. Using statement 3. of **Lemma D.4.1**, that states that sublevel sets of the relative entropies are compact in $\mathcal{M}(\mathcal{X}_T)$ for the topology of the weak convergence, this implies the existence of a subsequence N^{φ_n} that converges in distribution to some probability measure. By *Skorokhod's representation theorem* (**Proposition C.1.6**) there exists a random variable N^* with this limit law as distribution.
If $\mathsf{law}(N^*)$ and $\mathsf{law}(N^\varphi)$ coincide we can apply **Proposition D.4.2** and pass to the limit in

$$-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] \leq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi^n) + F(N^{\varphi^n})]$$

which finishes the proof of the theorem.

For convenience we introduce the following notation,

$$\langle f, N^\varphi \rangle(\bar{m}) := \int_{\mathcal{Y}_T} \mathbf{1}_{[0,\varphi(r)]} f(t,x) \bar{N}(dt,dx,dr)(\bar{m}) \quad \text{for } f \in C_c(\mathcal{Y}_T), \bar{m} \in \mathcal{M}(\mathcal{Y}_T).$$

In order to show $\mathsf{law}(N^*) = \mathsf{law}(N^\varphi)$ it is sufficient to show that for every $f \in C_c(\mathcal{Y}_T)$

$$\lim_{n\to\infty} \langle f, N^{\varphi_n} \rangle = \langle f, N^\varphi \rangle \quad \text{in } L^1(\bar{\mathbb{P}}). \tag{D.5.16}$$

Claim D.5.1. *N^* has the same distribution as N^φ.*

Let n_0 be large enough such that $\operatorname{supp}(f)$ is contained in $[0,T] \times K_{n_0}$. Then for all $n \geq n_0$, we have

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[|\langle f, N^{\varphi^n}\rangle - \langle f, N^\varphi\rangle|\Big] \\
&= \bar{\mathbb{E}}\Big[\Big|\int |f(y)(\mathbf{1}_{[0,\varphi]}(r) - \mathbf{1}_{[0,\varphi^n]}(r))|\bar{N}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}r)\Big|\Big] \\
&= \bar{\mathbb{E}}\Big[\int |f(y)(\mathbf{1}_{[0,\varphi]}(r) - \mathbf{1}_{[0,\varphi_n]}(r))|\bar{\nu}_T(\mathrm{d}s,\mathrm{d}y,\mathrm{d}r)\Big] \\
&\leq ||f||_\infty \bar{\mathbb{E}}\Big[\int_{K_{n_0}\times[0,T]\times\mathcal{M}(\mathcal{Y}_T)} |(\mathbf{1}_{[0,\varphi]}(r) - \mathbf{1}_{[0,\varphi_n]}(r))|\bar{\nu}_T(\mathrm{d}s,\mathrm{d}y,\mathrm{d}r)\Big].
\end{aligned}$$

Since we have for all $n \in \mathbb{N}$ and $r \in [0, \infty)$

$$|\mathbf{1}_{[0,\varphi]}(r) - \mathbf{1}_{[0,\varphi^n]}(r))| \leq \mathbf{1}_{[\varphi,\frac{1}{n}]}(r)\mathbf{1}_{\{\varphi<\frac{1}{n}\}}(r) + \mathbf{1}_{[\varphi,n]}(r)\mathbf{1}_{\{\varphi>n\}}(r),$$

it follows

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\Big|\langle f, N^{\varphi^n}\rangle - \langle f, N^{\varphi}\rangle\Big|\Big] \\
&\leq |f|_\infty \bar{\mathbb{E}}\Big[\int_{[0,T]\times K_{n_0}\times\mathcal{M}(\mathcal{Y}_T)} \Big(\frac{1}{n}\mathbf{1}_{\{\varphi<\frac{1}{n}\}} + (\varphi(s,x) - n)^+\mathbf{1}_{\{\varphi>n\}}\Big)\bar{\nu}_T(\mathrm{d}s, \mathrm{d}x.\mathrm{d}r)\Big].
\end{aligned}$$

We note that $\nu_T([0,T] \times K_{n_0}) < \infty$. Therefore the integral of the first part of the integrand converges to 0.
For the second integrand we obtain $(\varphi(t,x) - n)^+ \leq \ell(\varphi(t,x))$, for $n \geq 2$, where ℓ is defined in (D.5.3). Using $\bar{\mathbb{E}}[\mathfrak{L}_T(\varphi)] < \infty$ and $(\varphi(s,x) - n)^+\mathbf{1}_{\{\varphi>n\}} \leq \ell(\varphi(s,x))$, By dominated convergence theorem it follows that the right hand side of the last expression tends to zero as $n \to \infty$, which concludes the proof.

□

We follow with the proof of the lower bound of the variational formula of **Theoren D.5.1**.

Theorem D.5.3. *For any $F \in M_b(\mathcal{M}(\mathcal{X}_T))$ we have*

$$-\ln \bar{\mathbb{E}}[e^{F(N^1)}] \geq \inf_{\varphi\in\bar{\mathcal{A}}^+} \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi) + F(N^{\varphi})].$$

Proof. Using **Lemma D.5.4**, we proceed by approximating a general function $F \in M_b(\mathcal{M}(\mathcal{X}_T))$ by cylindrical functions.
Using the variational formula of Laplace functionals in terms of relative entropies stated in **Theorem D.4.1** we obtain

$$-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] = R(\bar{\mathbb{Q}}||\bar{\mathbb{P}}) + \bar{\mathbb{E}}^{\bar{\mathbb{Q}}}[F(h(\bar{N}))],$$

where $\bar{\mathbb{Q}}$ is the probability measure defined by

$$\bar{\mathbb{Q}}(A) := \frac{\int_A e^{-F(h(\bar{m}))}d\bar{\mathbb{P}}(\bar{m})}{\int_{\mathcal{M}(\mathcal{Y}_T)} e^{-F(h(\bar{m}))}d\bar{\mathbb{P}}(\bar{m})}, \quad A \in \mathcal{B}(\mathcal{M}(\mathcal{Y}_T)) \tag{D.5.17}$$

Using the *martingale representation theorem for Poisson random measures* (see *Jacod and Shyriaev (1987)*-**Section 4**) there exists an $(\bar{\mathcal{F}}_t)_{t\geq0}$ predictable process $\bar{\varphi}$ such that

$$\frac{d\bar{\mathbb{Q}}}{d\bar{\mathbb{P}}} = \mathcal{E}_T(\bar{\varphi}), \tag{D.5.18}$$

where $\mathcal{E}_T$ is the Doleans-Dade exponential defined in (D.5.5). Since $F \in C_{cyl}(\mathcal{M}(\mathcal{X}_T))$ and $\bar{\varphi} \in \mathcal{A}_{b,n}$, for some $n \in \mathbb{N}$, we know from the relative entropy representation for the Laplace transform of $F(N^1)$ (**Theorem D.4.1**) that

$$-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] = \bar{\mathbb{E}}^{\bar{\mathbb{Q}}}[\mathfrak{L}_T(\bar{\varphi}) + F(h(\bar{N}))]. \tag{D.5.19}$$

In order to stress the dependence of $\bar{\mathbb{Q}}$ in terms of $\bar{\varphi}$ we write $\bar{\mathbb{Q}}$ as $\bar{\mathbb{Q}}^{\bar{\varphi}}$. Given F cylindrical, it remains to construct a near minimizer on the original probability space. Fix $\delta \in (0,1)$. We construct $\varphi \in \bar{\mathcal{A}}^+_{s,n}$ such that

$$\bar{\mathbb{E}}^{\bar{\mathbb{Q}}^{\bar{\varphi}}}[\mathfrak{L}_T(\bar{\varphi}) + F(h(\bar{N}))] \geq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi) + F(N^{\varphi})] - \delta. \tag{D.5.20}$$

Since δ is arbitrary, the proof is complete for the class of the functions $C_{cyl}(\mathcal{M}(\mathcal{X}_T))$ if we construct such $\varphi \in \bar{\mathcal{A}}^+_{s,n}$.
Fix $\delta \in (0,1)$and let $\bar{\varphi}_k$ be a sequence in $\bar{\mathcal{A}}^+_s$ as in the **Lemma D.5.2** for the function $\bar{\varphi}$ that is stated in (D.5.5).

Claim D.5.2. $\bar{\mathbb{E}}^{\bar{\mathbb{Q}}^{\bar{\varphi}_k}}[\mathcal{L}_T(\bar{\varphi}_k) + F(h(\bar{N}))] \to \bar{\mathbb{E}}^{\bar{\mathbb{Q}}^{\bar{\varphi}}}[\mathcal{L}_T(\bar{\varphi}) + F(h(\bar{N}))]$

Rewriting the last expression in terms of the original probability measure $\bar{\mathbb{P}}$ we derive

$$\begin{aligned}
&\bar{\mathbb{E}}^{\bar{\mathbb{Q}}^{\bar{\varphi}_k}}[\mathfrak{L}_T(\bar{\varphi}_k) + F(h(\bar{N}))] = \bar{\mathbb{E}}[\mathcal{E}_T(\bar{\varphi}_k)(\mathfrak{L}_T(\bar{\varphi}_k) + F(h(\bar{N})))] \text{ and}\\
&\bar{\mathbb{E}}^{\bar{\mathbb{Q}}^{\bar{\varphi}}}[\mathfrak{L}_T(\bar{\varphi}) + F(h(\bar{N}))] = \bar{\mathbb{E}}[\mathcal{E}_T(\bar{\varphi})(\mathfrak{L}_T(\bar{\varphi}) + F(h(\bar{N})))].
\end{aligned}$$

To show the claim, it is enough to see that

$$\bar{\mathbb{E}}[(\mathcal{E}_T(\bar{\varphi}_k) - \mathcal{E}_T(\bar{\varphi}))(\mathfrak{L}_T(\bar{\varphi}_k) + F(h(\bar{N})))] \to 0 \text{ as } k \to \infty. \tag{D.5.21}$$

Lemma D.5.2 assures the existence of an approximation sequence in the class of nice controls $\bar{\mathcal{A}}^+_{s,n}$. We observe that F is bounded and hence $\mathcal{E}_T(\bar{\varphi})$ turns itself to be bounded since $(\bar{\varphi}_k)_{k\in\mathbb{N}} \subset \mathcal{A}_{b,n}$.
The functionals $(\mathfrak{L}_T(\bar{\varphi}_k))_{k\in\mathbb{N}}$ are uniformly bounded for the same reason $((\bar{\varphi}_k)_{k\in\mathbb{N}} \subset \mathcal{A}_{b,n})$.
We conclude from statement 3. of **Lemma D.5.2** that we can perform the limits in (D.5.21). Same **Lemma D.5.2** assures that

$$\bar{\mathbb{E}}[\mathcal{E}_T(\bar{\varphi})(\mathfrak{L}_T(\bar{\varphi}_k) - \mathfrak{L}_T(\bar{\varphi}))] \to 0 \quad \text{as } k \to \infty. \tag{D.5.22}$$

Due to (D.5.21) and (D.5.22) we can pass to the limit in the next expression,

$$\begin{aligned}
&\bar{\mathbb{E}}[\mathcal{E}_T(\bar{\varphi}_k)(\mathfrak{L}_T(\bar{\varphi}_k) + F(h(\bar{N}))] - \bar{\mathbb{E}}[\mathcal{E}_T(\bar{\varphi})(\mathfrak{L}_T(\bar{\varphi}) + F(h(\bar{N}))]\\
&= \bar{\mathbb{E}}[F(h(\bar{N}))(\mathcal{E}_T(\bar{\varphi}_k) - \mathcal{E}_T(\bar{\varphi}))] + \bar{\mathbb{E}}[\mathcal{E}_T(\bar{\varphi}_k)(\mathfrak{L}_T(\bar{\varphi}_k) - \mathfrak{L}_T(\bar{\varphi}))]\\
&+ \bar{\mathbb{E}}[\mathfrak{L}_T(\bar{\varphi})(\mathcal{E}_T(\bar{\varphi}_k) - \mathcal{E}_T(\bar{\varphi}))]
\end{aligned}$$

and the claim follows.

Fix now k large enough such that the difference between the two sides in the statement of **Claim D.5.2** is bounded by δ.

According to the second statement of **Lemma D.5.3** we can find $\varphi \in \bar{\mathcal{A}}_s^+$ such that (D.5.20) is satisfied. This proves the theorem when $F \in C_{cyl}(\mathcal{M}(\mathcal{X}_T))$.
We consider now a general $F \in \mathcal{M}_b(\mathcal{M}(\mathcal{X}_T))$. By density there exists a sequence $(F_j)_{j\in\mathbb{N}} \in C_{cyl}(\mathcal{M}(\mathcal{X}_T))$ such that $||F_j||_\infty \leq ||F||_\infty < \infty$ and $F_j \to F$ $\mathbb{P}$- a.s. as $j \to \infty$ on $\mathcal{M}(\mathcal{X}_T)$. Using dominated convergence

$$-\ln \bar{\mathbb{E}}[e^{-F_j(N^1)}] \to -\ln \bar{\mathbb{E}}[e^{-F(N^1)}] \quad \text{as } j \to \infty.$$

Fix $j \in \mathbb{N}$ and let $\bar{\varphi}^j \in \bar{\mathcal{A}}_{b,n}^+$ be determined by the martingale representation (see Chapter 4-section 3 in *Jacod Shiryaev (1987)*-Chapter 4 (Section 3)) applied to the respective density $\frac{d\bar{\mathbb{Q}}}{d\bar{\mathbb{P}}}$, where $\bar{\mathbb{Q}}$ is defined as before in (D.5.17). Let $(\varphi_k^j)_{k\in\mathbb{N}} \subset \bar{\mathcal{A}}_{s,n}^+$ a sequence of simple controls that approximates $\bar{\varphi}_j$ such as in **Lemma D.5.2** . Given $\delta \in (0,1)$, let $\varphi_k^j \in \bar{\mathcal{A}}_{s,n}^+$ such that

$$\bar{\mathbb{E}}^{\bar{\mathbb{Q}}^{\bar{\varphi}_j}}[\mathfrak{L}_T(\bar{\varphi}_j) + F(h(\bar{N}))] \geq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_k^j) + F(N^{\varphi_k^j})] - \delta. \tag{D.5.23}$$

The preceeding inequality and **Lemma D.4.1**, which ensures

$$-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] = \bar{\mathbb{E}}^{\bar{\mathbb{Q}}^{\bar{\varphi}_j}}[\mathfrak{L}_T(\bar{\varphi}_j) + F(h(\bar{N}))],$$

implies that

$$\sup_{j\in\mathbb{N}} \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_j)] \leq 2||F||_\infty + 1$$

for any diagonal subsequence $(\varphi_j)_{j\in\mathbb{N}} \supset (\varphi_j^k)_{j,k\in\mathbb{N}}$.
As noted in (D.5.13) for all $j \in \mathbb{N}$ we have

$$R(\bar{\mathbb{P}} \circ (N^{\varphi_j})^{-1} || \bar{\mathbb{P}} \circ (N^1)^{-1}) \leq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_j)] \leq \sup_{||\psi||_\infty \leq n} \mathfrak{L}_T(\Phi) < \infty.$$

Since we have the uniform bound for the sub-level sets of the relative entropies above we extract a subsequence of the sequence N^{φ_j}, that we denote for sake of readibility N^{φ_j} convergent to a weak limit. Due to *Skorokhod's representation theorem* (**Proposition C.1.7**) this weak limit is the law of a certain random measure $N^* \in \mathcal{M}(\mathcal{X}_T)$ such that $N^{\varphi_j} \Rightarrow N^*$ as $j \to \infty$. In particular, using **Proposition D.4.2**, we infer

$$\bar{\mathbb{E}}[F_j(N^{\varphi_j})] \to \bar{\mathbb{E}}[F(N^*)]. \tag{D.5.24}$$

Using the duality result (D.5.8) and (D.5.20), since

$$-\ln \bar{\mathbb{E}}[e^{-F_j(N^1)}] \to -\ln \bar{\mathbb{E}}[e^{-F(N^1)}],$$

we have

$$\bar{\mathbb{E}}^{\mathbb{Q}^{\bar{\varphi}_j}}[\mathfrak{L}_T(\bar{\varphi}_j) + F(h(\bar{N}))] \geq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_j) + F(N^{\varphi_j})] - \delta.$$

For $j \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
-\ln \bar{\mathbb{E}}[e^{-F(N^1)}] &\geq -\ln \bar{\mathbb{E}}[e^{-F_j(N^1)}] - \delta \\
&= \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_j) + F_j(N^{\varphi_j})] - 2\delta \\
&\geq \bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_j) + F(N^{\varphi_j})] - 3\delta,
\end{aligned}$$

which finishes the proof.

□

D.6 A sufficient condition for a large deviations principle- Proof of Theorem 2.1.1

Proof. Consider $(\mathcal{G}^\varepsilon)_{\varepsilon>0}$, with

$$\mathcal{G}^\varepsilon : \mathbb{M} \to \mathcal{D},$$

satisfying **Condition 2.1.1** and $(Z^\varepsilon)_{\varepsilon>0}$ $\mathcal{D}$-valued random variables defined on the probability space $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}})$ by

$$Z^\varepsilon := \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}}),$$

for every $\varepsilon > 0$ We show that the family $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in the space $\mathcal{D}$, with good rate function $\mathbb{J}$ and speed $\varepsilon > 0$.

1. We prove that $\mathbb{J}$ is a good rate function. It suffices to show that, for any $a \in (0,\infty)$, the set

   $$\Lambda_a := \{\varphi \in \mathcal{D} \mid \mathbb{J}(\varphi) \le a\} \subset \mathcal{D}$$

   is compact.
   Fix $a \in (0,\infty)$. **Condition 2.1.1**(i) yields that, for any $M > 0$, the image set

   $$\Gamma_M := \{\mathcal{G}^0(\nu_T^g) \mid g \in S^M\} \subset \mathcal{D}$$

   is compact., since S^M is compact. The compactness of Λ_a is a consequence of

   $$\Lambda_a = \bigcap_{n\ge 1} \Gamma_{a+\frac{1}{n}}.$$

2. The proof that $(Z^\varepsilon)_{\varepsilon>0}$ satisfies a Laplace principle on $\mathcal{D}$ with the good rate function $\mathbb{J}$ uses **Theorem D.3.2**. We show that, for any $F \in C_b(\mathcal{D})$, it follows

   $$\lim_{\varepsilon\to 0} -\varepsilon \ln \bar{\mathbb{E}}\Big[e^{-\varepsilon^{-1}F(Z^\varepsilon)}\Big] = \inf_{\varphi\in\mathcal{D}} \Big[\mathbb{J}(\varphi) + F(\varphi)\Big]. \tag{D.6.1}$$

   We note that

   $$-\varepsilon \ln \bar{\mathbb{E}}\Big[e^{-\varepsilon^{-1}F(Z^\varepsilon)}\Big] = -\varepsilon \ln \bar{\mathbb{E}}\Big[e^{-\varepsilon^{-1}F\circ\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})}\Big].$$

   Since $N^{\varepsilon^{-1}}$ is a Poisson random measure with intensity $\varepsilon^{-1}ds \otimes \nu$ we conclude from **Theorem D.5.1** that

   $$-\varepsilon \ln \bar{\mathbb{E}}\Big[e^{-\varepsilon^{-1}F\circ\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})}\Big] = \inf_{\varphi\in\bar{\mathcal{A}}^+} \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi) + F\circ\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi})\Big].$$

   Given $\varepsilon \in (0,1)$ we fix $\delta_0 = \delta_0(\varepsilon) \in (0,1)$ such that for all $\delta^\varepsilon < \delta_0$ there exists a family $(\varphi_\varepsilon^\delta)_{\delta>0} \subset \bar{\mathcal{A}}^+$ satisfying

   $$\inf_{\varphi\in\bar{\mathcal{A}}^+} \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi) + F\circ\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi})\Big] \ge \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_\varepsilon^{\delta^\varepsilon}) + F\circ\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon^{\delta^\varepsilon}})\Big] - \delta. \tag{D.6.2}$$

Given $\varepsilon \in (0,1)$, we fix $\delta^\varepsilon < \delta_0^\varepsilon$ and, in a slight abuse of notation, we write the so constructed sequence $(\varphi_\varepsilon^{\delta^\varepsilon})_{\varepsilon>0}$ by $(\varphi_\varepsilon)_{\varepsilon>0}$.
Note that (D.6.2) implies $\bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_\varepsilon)] \leq 2||F||_\infty + \delta$.

We set, for $t \leq 0 \leq T$

$$\mathfrak{L}_t(\varphi_\varepsilon) := \int_0^t \int_{\mathbb{R}^d} \ell(\varphi_\varepsilon(s,z))\nu(dz)ds.$$

Given $M \geq 0$ we define, with the convention that the infimum of the empty set is ∞,

$$\begin{aligned} &\tau_M^\varepsilon = \inf\{t \geq 0 : \mathfrak{L}_t(\varphi_\varepsilon) \geq M\} \wedge T \quad \text{ and} \\ &\varphi_{\varepsilon,M}(t,z) = 1 + [\varphi_\varepsilon(t,z) - 1]\mathbf{1}_{[0,\tau_M^\varepsilon]}(t). \end{aligned}$$

Note that $\varphi_{\varepsilon,M} \in \mathcal{U}_+^M$ $\bar{\mathbb{P}}$- a.s.
Moreover, due to *Chebyshev's inequality* (**Proposition B.3.3**) it follows

$$\begin{aligned} \bar{\mathbb{P}}(\varphi_\varepsilon \neq \varphi_{\varepsilon,M}) &\leq \bar{\mathbb{P}}(\mathfrak{L}_T(\varphi_\varepsilon) \geq M) \\ &\leq \frac{\bar{\mathbb{E}}[\mathfrak{L}_T(\varphi_\varepsilon)]}{M} \\ &\leq \frac{2||F||_\infty + \delta}{M}. \end{aligned}$$

Hence,

$$\begin{aligned} &\bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_\varepsilon) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})\Big] - \delta \\ &\geq \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_{\varepsilon,M}) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_{\varepsilon,M}})\Big]\bar{\mathbb{P}}(\varphi_\varepsilon = \varphi_{M,\varepsilon}) - \delta \\ &= \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_{\varepsilon,M}) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_{\varepsilon,M}})\Big] \\ &- \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_\varepsilon) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})\Big]\bar{\mathbb{P}}(\varphi_\varepsilon \neq \varphi_{M,\varepsilon}) - \delta \\ &\geq \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_{\varepsilon,M}) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_{\varepsilon,M}})\Big] \\ &- \frac{(3||F||_\infty + \delta)(2||F||_\infty + 1)}{M} - \delta \\ &\geq \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_{\varepsilon,M}) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_{\varepsilon,M}})\Big] - 2\delta. \end{aligned}$$

We choose $M > 0$ large enough such that

$$\frac{(3||F||_\infty + \delta)(2||F||_\infty + \delta)}{M} < \delta.$$

Note that $(\varphi_{\varepsilon,M})$ is a family of S^M valued random variables and recalling that S^M is compact we choose a weakly convergent subsequence. Let us denote φ the weak limit. From part 2 of **Condition 2.1.1**, we have that, along this subsequence, $\mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}\varphi_{\varepsilon,M}})$ converges weakly to $\mathcal{G}^{0}(\nu_T^{\varphi})$. Using *Fatou's lemma* and the lower semi-continuity of the relative entropy functional, we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \varepsilon \ln \bar{\mathbb{E}}\Big[e^{-\varepsilon^{-1}F(Z^{\varepsilon})}\Big] &\geq \liminf_{\varepsilon\to 0} \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi_{\varepsilon}) + F\circ\mathcal{G}^{\varepsilon}\Big(\varepsilon N^{-1}\varphi_{\varepsilon,M}\Big)\Big] - 2\delta \\
&\geq \bar{\mathbb{E}}\Big[\mathfrak{L}_T(\varphi) + F\circ\mathcal{G}^{0}(\nu_T^{\varphi})\Big] - 2\delta \\
&\geq \inf_{q\in\mathcal{D}} \mathfrak{L}_T(q) - 2\delta.
\end{aligned}$$

We proceed from a reverse inequality that concludes the proof of (D.6.1). Fix δ arbitrary and choose $\varphi_0 \in \mathcal{D}$ such that

$$\mathbb{J}(\varphi_0) + F(\varphi_0) \leq \inf_{\varphi\in\mathcal{D}} (\mathbb{J}(\varphi) + F(\varphi)) + \delta.$$

Choose $g \in \mathbb{S}_{\varphi_0}$ such that $\mathfrak{L}_T(g) \leq \mathbb{J}(\varphi_0) + \delta$. We note that with this choice we have

$$\varphi_0 = \mathcal{G}^{0}(\nu_T^{g}).$$

Recalling an argument similar as the one used to obtain (D.6.2) we derive

$$\begin{aligned}
\limsup_{\varepsilon\to 0} -\varepsilon \ln \bar{\mathbb{E}}[\varepsilon^{-\varepsilon^{-1}F(Z^{\varepsilon})}] &\leq \mathfrak{L}_T(g) + \limsup_{\varepsilon\to 0} \bar{\mathbb{E}}\Big[F\circ\mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}})\Big] \\
&\leq \mathbb{J}(\varphi_0) + \delta + F\circ\mathcal{G}^{0}(\nu_T^{g}) \\
&= \mathbb{J}(\varphi_0) + F(\varphi_0) + \delta \\
&\leq \inf_{\varphi\in\mathcal{D}} (\mathbb{J}(\varphi) + F(\varphi)) + 2\delta.
\end{aligned}$$

We note that the second inequality makes use of part 2 of **Condition 2.1.1**. Since δ is arbitrary the proof is concluded.

$\square$

D.7 A sufficient condition for a moderate deviations principle- Proof of Theorem 3.1.1

In order to prove **Theorem 3.1.1** we introduce some technical lemmas. The first is a very useful lemma concerning numerical inequalities that is used more than once in this thesis.

Lemma D.7.1.

a) For $a, b > 0$ and $\sigma \geq 1$, we have

$$ab \leq e^{\sigma a} + \frac{1}{\sigma}\ell(b). \tag{D.7.1}$$

b) For every $\beta > 0$, there exists $\kappa_1(\beta), \kappa_1'(\beta) > 0$ such that $\kappa(\beta), \kappa_1'(\beta) \to 0$ as $\beta \to +\infty$ and

$$|x-1| \leq \kappa_1(\beta)\ell(x) \text{ for } |x-1| \geq \beta,\ x \geq 0 \quad \text{ and } x \leq \kappa_1'(\beta)\ell(x), \text{ for } x \geq \beta.$$

c) For each $\beta > 0$, there exists $\kappa_2(\beta) > 0$ such that

$$|x-1|^2 \leq \kappa_2(\beta)\ell(x) \text{ for } |x-1| \leq \beta \text{ and } x \geq 0.$$

d) There exists $\kappa_3(\beta) > 0$ such that

$$\ell(x) \leq \kappa_3(\beta)|x-1|^2 \text{and } \left|\ell(x) - \frac{(x-1)^2}{2}\right| \leq \kappa_3(\beta)|x-1|^3 \quad \text{for } x \geq 0.$$

We can assume without loss of generality that $\kappa_2(\beta)$ is non-increasing in β.

Proof. 1. We start to prove statement (a) of the lemma. We consider $f(x) = e^x, \quad x \geq 0$. The *Fenchel-Legendre convex conjugate* of f is the function

$$f^*(y) = y\ln y - y, \quad y \geq 0.$$

If we consider $\tilde{f}(x) = e^{\sigma x} = f(\sigma x), \quad$ for $\sigma > 1$ using the properties of *Fenchel-Legendre's convex conjugates* it follows

$$\tilde{f}^*(y) = f^*(\frac{y}{\sigma}) = \frac{y}{\sigma}\ln\frac{y}{\sigma} - \frac{y}{\sigma},$$

and **Young-Legendre's inequality** reads as

$$\begin{aligned} ab &\leq \tilde{f}(a) + \tilde{f}^*(b) \\ &= e^{\sigma a} + \frac{b}{\sigma}\ln\frac{b}{\sigma} - \frac{b}{\sigma} \\ &= e^{\sigma a} + \frac{1}{\sigma}(b\ln b - b) - \frac{\ln\sigma}{\sigma} \\ &\leq e^{\sigma a} + \frac{1}{\sigma}\ell(b), \quad \text{for all } a, b \geq 0, \sigma > 1, \end{aligned}$$

which finishes the proof statement 1.

2. Statements (b), (c). and (d) follows from similar argument that we use to prove the following fact:

$$\text{For all } \beta > 0, \text{ there exists } \kappa_1(\beta) > 0, \text{ such that}$$
$$\lim_{\beta\to\infty} \kappa_1(\beta) = 0 \text{ and } x \leq k_1(\beta)\ell(x).$$

We define the auxiliary function $\psi(x) = \ell(x) - Ax$, for all $x \geq 0$ and for some $A > 0$ that will be fixed later. Computing the derivative of ψ and analysing its signal we conclude that

$$\psi'(x) = \ln x + 1 - 1 - A = \ln x - A$$
$$\psi'(x) \geq 0 \Leftrightarrow x \geq e^A,$$
$$\psi(x) \geq \psi(e^A) \text{ and}$$
$$\psi \nearrow \text{ in } [e^A, \infty).$$

Since $\psi(e^A) = 1 - e^A < 0$ $(A > 0)$ and $\lim_{x\to\infty} \psi(x) = \infty$, it follows from intermediate-value theorem for continuous functions that there exists $x_A \geq e^A$ such that $\psi(x) \geq 0$, for all $x \geq x_A$.
We define $\beta = e^A$. Then for all $x \geq x_\beta \geq \beta$, $\psi(x) \geq 0$ or equivalently

$$x \leq \frac{1}{\ln\beta}\ell(x), \qquad \text{for all } x \geq \beta.$$

Defining $\kappa_1(\beta) = \frac{1}{\beta}$ the result follows.

□

Lemma D.7.2. *Suppose $\varphi \in \mathcal{S}^M_{+,\varepsilon}$ for some $M < \infty$. Set $\psi = \frac{\varphi-1}{a(\varepsilon)}$. Then for all $\beta > 0$ we have*

a) $\displaystyle\int_{\mathbb{R}^d\times[0,T]} |\psi| 1_{\{\psi|\geq\beta/a(\varepsilon)\}} d\nu_T \leq Ma(\varepsilon)\kappa_1(\beta)$;

b) $\displaystyle\int_{\mathbb{R}^d\times[0,T]} \varphi 1_{\{|\varphi|>\beta\}} d\nu_T \leq Ma^2(\varepsilon)\kappa_1'(\beta)$;

c) $\displaystyle\int_{\mathbb{R}^d\times[0,T]} |\psi|^2 1_{\{|\psi|\leq\beta/a(\varepsilon)\}} d\nu_T \leq Mk_2(\beta)$.

Proof. The proof follows immediately from **Lemma D.7.1**.

1. In order to prove a) we observe that $|\psi| \geq \frac{\beta}{a(\varepsilon)}$ is equivalent to $|\varphi - 1| \leq \beta$ and by the statement (b) of previous lemma, we have that $|\varphi - 1| \leq k_1(\beta)\ell(\varphi)$. It follows from that and from the fact $\mathfrak{L}_T(\varphi) \leq Ma^2(\varepsilon)$,

$$\int_{\mathbb{R}^d\times[0,T]} |\psi|\mathbf{1}_{\{\psi|\geq\beta/a(\varepsilon)\}} d\nu_T \leq \int_{\mathbb{R}^d\times[0,T]} k_1(\beta)\frac{\ell(\varphi)}{a(\varepsilon)} d\nu_T$$
$$\leq Ma(\varepsilon)k_1(\beta).$$

2. The proof of statement (b) follows from applying (c) from previous lemma and observing again that $\mathfrak{L}_T(\varphi) \leq Ma^2(\varepsilon)$.

3. Statement (c) follows from statement (c) from previous lemma and from the fact $\mathfrak{L}_T(\varphi) \leq Ma^2(\varepsilon)$. □

We follow with the proof of **Theorem 3.1.1**

Proof of Theorem 3.1.1.

1. We prove that I is a good rate function:
   Statement (i) of **Condition 3.1.1** implies that $\Gamma_K := \{g \in B_2(K)\}$ is compact for all $K < \infty$ and we observe that, given $M < \infty$, a sublevel set of $\mathbb{I}$ is of the form

$$\{\eta \in \mathbb{U} : \mathbb{I}(\eta) \leq M\} = \bigcap_{n \geq 1} \Gamma_{2M+\frac{1}{n}},$$

   which proves that the sublevel sets of I are compact, since the intersection of compacts is compact.

2. In order to prove that $(Y^\varepsilon)_{\varepsilon>0} := (\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}}))_{\varepsilon>0}$ satisfies a large deviations principle with speed $b(\varepsilon)$ and good rate function $\mathbb{I}$, due to **Theorem D.3.2**, it suffices to show the Laplace principle upper and lower bounds,
   for all $F \in C_b(\mathbb{U})$,

$$\limsup_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}F(Z^\varepsilon)}\Big] \leq -\inf_{x \in \mathcal{D}}\{h(x) + \mathbb{I}(x)\},$$
$$\liminf_{\varepsilon \to 0} b(\varepsilon) \ln \mathbb{E}\Big[e^{-\frac{1}{b(\varepsilon)}F(Z^\varepsilon)}\Big] \geq -\inf_{x \in \mathcal{D}}\{h(x) + \mathbb{I}(x)\}.$$

   Using **Theorem D.5.1**, we have that

$$-b(\varepsilon) \ln \bar{\mathbb{E}}\Big[e^{-\frac{F(Y^\varepsilon)}{b(\varepsilon)}}\Big] = \inf_{\varphi \in \bar{\mathcal{A}}_+} \bar{\mathbb{E}}\Big[\frac{b(\varepsilon)}{\varepsilon}\mathfrak{L}_T(\varphi) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi})\Big]. \tag{D.7.2}$$

   We prove first the lower bound,

$$\liminf_{\varepsilon \to 0} -b(\varepsilon) \ln \bar{\mathbb{E}}\Big[e^{-\frac{F(Z^\varepsilon)}{b(\varepsilon)}}\Big] \geq \inf_{\eta \in \mathcal{D}}\{\mathbb{I}(\eta) + F(\eta)\}. \tag{D.7.3}$$

   Given $\varepsilon > 0$ choose $\tilde{\varphi}^\varepsilon \in \bar{\mathcal{A}}_b^+$ such that

$$-b(\varepsilon) \ln \bar{\mathbb{E}}[e^{-\frac{F(Z^\varepsilon)}{b(\varepsilon)}}] \geq \bar{\mathbb{E}}\Big[\frac{b(\varepsilon)}{\varepsilon}\mathfrak{L}_T(\tilde{\varphi}^\varepsilon) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\tilde{\varphi}^\varepsilon})\Big]. \tag{D.7.4}$$

   Since $||F||_\infty < \infty$ and, for all $\varepsilon \in (0,1)$ assuming that $b(\varepsilon) \leq 1$, it follows

$$\infty > C := (||F||_\infty + 1) \geq \bar{\mathbb{E}}\Big[\frac{b(\varepsilon)}{\varepsilon}\mathfrak{L}_T(\tilde{\varphi}^\varepsilon)\Big]. \tag{D.7.5}$$

We set for $0 \le t \le T$

$$L_t(\tilde{\varphi}^\varepsilon) := \int_0^t \int_{\mathbb{R}^d} \ell(\tilde{\varphi}^\varepsilon(s,z))\nu(dz)ds.$$

We fix $\delta > 0$ and define the stopping time

$$\tau^\varepsilon := \inf\Big\{t \in [0,T] : \frac{b(\varepsilon)}{\varepsilon}\mathfrak{L}_t(\tilde{\varphi}^\varepsilon) > \frac{2C||F||_\infty}{\delta}\Big\} \wedge T.$$

Next, we construct the control $\varphi^\varepsilon \in \bar{\mathcal{A}}_b^+$ as follows,

$$\varphi^\varepsilon(s,z) = \tilde{\varphi}^\varepsilon(s,z)\mathbf{1}_{\{s \le \tau^\varepsilon\}} + \mathbf{1}_{\{s > \tau^\varepsilon\}}, \quad (s,z) \in [0,T] \times \mathbb{R}^d.$$

By construction, it follows that

$$\frac{b(\varepsilon)}{\varepsilon}\mathcal{L}_T(\varphi^\varepsilon) \le \tilde{C} := \frac{2C||F||_\infty}{\delta}.$$

Also due to (D.7.7) it follows that

$$\begin{aligned} \bar{\mathbb{P}}(\varphi^\varepsilon \neq \tilde{\varphi}^\varepsilon) &\le \bar{\mathbb{P}}\Big(\frac{b(\varepsilon)}{\varepsilon}\mathfrak{L}_T(\tilde{\varphi}^\varepsilon) > \bar{C}\Big) \\ &\le \frac{\bar{\mathbb{E}}\Big[\frac{b(\varepsilon)}{\varepsilon}\mathfrak{L}_T(\tilde{\varphi}^\varepsilon)\Big]}{\bar{C}} \\ &\le \frac{\delta}{2||F||_\infty}. \end{aligned} \tag{D.7.6}$$

For all $(z,s) \in [0,T] \times \mathbb{R}^d$ we define the centered normalized controls,

$$\tilde{\psi}^\varepsilon(s,z) := \frac{\bar{\varphi}^\varepsilon(s,z) - 1}{a(\varepsilon)}, \quad \psi^\varepsilon(s,z) := \frac{\varphi^\varepsilon(s,z) - 1}{a(\varepsilon)} \equiv \tilde{\psi}^\varepsilon(s,z)\mathbf{1}_{\{s \le \tau^\varepsilon\}}.$$

Fix $\beta \in (0,1]$ and let $\beta_\varepsilon = \frac{\beta}{a(\varepsilon)}$. In the next estimates we use (D.7.4) in the first line, statement (d) of **Lemma D.7.1**, statement (d) of **Lemma D.7.2** and the observation that $\kappa_2(1) \ge \kappa_2(\beta)$,

$$\begin{aligned} -b(\varepsilon)\bar{\mathbb{E}}[e^{-\frac{F(Z^\varepsilon)}{b(\varepsilon)}}] &\ge \bar{\mathbb{E}}\Big[\frac{b(\varepsilon)}{\varepsilon}\int_{[0,T]\times\mathbb{R}^d} \ell(\bar{\varphi}^\varepsilon)d\nu_T + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\bar{\varphi}^\varepsilon})\Big] - \varepsilon \\ &\ge \bar{\mathbb{E}}\Big[\int_{[0,T]\times\mathbb{R}^d} \frac{b(\varepsilon)}{\varepsilon}\ell(\varphi^\varepsilon)\mathbf{1}_{\{|\psi^\varepsilon| \le \beta_\varepsilon\}}d\nu_T + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\bar{\varphi}^\varepsilon})\Big] - \varepsilon \\ &\ge \bar{\mathbb{E}}\Big[\frac{1}{2}((\psi^\varepsilon)^2 - k_3 a(\varepsilon)|\psi^\varepsilon|^3)\mathbf{1}_{\{|\psi^\varepsilon| \le \beta_\varepsilon\}}d\nu_T\Big] \\ &\quad + \bar{\mathbb{E}}\Big[F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\tilde{\varphi}^\varepsilon}) - F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon}))\Big] \\ &\ge \bar{\mathbb{E}}\Big[\frac{1}{2}\int_{[0,T]\times\mathbb{R}^d} (\psi^\varepsilon)^2\mathbf{1}_{\{|\psi|^\varepsilon \le \beta_\varepsilon\}} + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon})\Big] - \delta \\ &\quad - \varepsilon - \frac{1}{2}\beta k_3 M \kappa_2(1). \end{aligned} \tag{D.7.7}$$

In the last inequality it was used the estimate combined with (D.7.6),

$$\left|\bar{\mathbb{E}}\Big[F\circ\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\tilde{\varphi}^\varepsilon}) - F\circ\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon})\Big]\right| \le 2|F|_\infty\bar{\mathbb{P}}(\varphi^\varepsilon \neq \tilde{\varphi}^\varepsilon) \le \delta.$$

Due to **Lemma D.7.2** and the monotonicity of $k_2(\beta)$, $(\psi^\varepsilon \mathbf{1}_{\{|\psi^\varepsilon|\le \frac{\beta}{a(\varepsilon)}\}})_{\varepsilon>0}$ is a family of $B_2(\sqrt{Mk_2(1)})$ - valued random variables. Using the weak compactness of $B_2(r)$, we can conclude by *Banach-Alaoglu theorem* (**Theorem A.1.1**) that $(\psi^\varepsilon \mathbf{1}_{\{|\psi^\varepsilon|\le \frac{\beta}{a(\varepsilon)}\}})_{\varepsilon>0}$ has a weak limit point. Therefore, let ψ be a limit point of ψ_ε. By contradiction it suffices to show (D.7.3) along a subnet with φ has a limit point. We denote such subnet, for sake or readiblity, as $(\psi_\varepsilon)_{\varepsilon>0}$. From condition (ii) of **Condition 3.1.1** along this family $\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon})$ converges in law to $\eta = \mathcal{G}^0$. Hence taking limits in (D.7.7) along this subnet, it follows, using *Fatou's lemma* and the definition of $\mathbb{I}$ given by (**3.1.3**).

$$\begin{aligned}\liminf_{\varepsilon\to 0} -b(\varepsilon)\bar{\mathbb{E}}[e^{-\frac{F(Z^\varepsilon)}{b(\varepsilon)}}] &\ge \bar{\mathbb{E}}\Big[\frac{1}{2}\int_{[0,T]\times\mathbb{R}^d}(\psi)^2 + F(\eta)\Big] - \delta - \frac{1}{2}\beta k_3 M\kappa_2(1)\\ &\ge \bar{\mathbb{E}}[\mathbb{I}(\eta)+F(\eta)] - \delta - \frac{1}{2}\beta k_3 M\kappa_2(1)\\ &\ge \inf_{\eta\in\mathcal{D}}[\mathbb{I}(\eta)+F(\eta)] - \delta - \frac{1}{2}\beta k_3 M\kappa_2(1).\end{aligned}$$

Sending $\delta,\beta\to 0$ we prove the desired lower bound (D.7.3).

We prove now the upper bound,

$$\limsup_{\varepsilon\to 0} -b(\varepsilon)\ln\bar{\mathbb{E}}\Big[e^{-\frac{F(Z^\varepsilon)}{b(\varepsilon)}}\Big] \le \inf_{\eta\in\mathcal{D}}[\mathbb{I}(\eta)+F(\eta)]. \tag{D.7.8}$$

Fix $\delta>0$ and let $\eta\in\mathcal{D}$ such that

$$\mathbb{I}(\eta)+F(\eta) \le \inf_{\eta\in\mathcal{D}}[\mathbb{I}(\eta)+F(\eta)] + \frac{\delta}{2}. \tag{D.7.9}$$

Choose $\psi\in L^2(\nu_T)$ such that

$$\frac{1}{2}\int_{[0,T]\times\mathbb{R}^d}|\psi|^2 d\nu_T \le \mathbb{I}(\eta)+\frac{\delta}{2}, \tag{D.7.10}$$

where $\eta = \mathcal{G}^0(\psi)$.
For $\beta\in(0,1]$ and $(s,z)\times[0,T]\times\mathbb{R}^d$ we define

$$\psi^\varepsilon(s,z) = \psi\mathbf{1}_{\{|\psi|\le\frac{\beta}{a(\varepsilon)}\}}(s,z) \quad\text{ and }\quad \varphi^\varepsilon(s,z) = 1 + a(\varepsilon)\psi^\varepsilon(s,z).$$

Lemma D.7.1 implies that, given $\varepsilon > 0$,

$$\begin{aligned}\int_{[0,T]\times\mathbb{R}^d} \ell(\varphi^\varepsilon)d\nu_T &\le k_3 \int_{[0,T]\times\mathbb{R}^d} (\varphi^\varepsilon - 1)^2 d\nu_T \\ &= a^2(\varepsilon)k_3 \int_{[0,T]\times\mathbb{R}^d} |\psi^\varepsilon|^2 d\nu_T \\ &\le a^2(\varepsilon)M,\end{aligned}$$

where $M = k_3 \int_{[0,T]\times\mathbb{R}^d} |\psi|^2 d\nu_T$. So $\varphi^\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$ for all $\varepsilon > 0$. Also the equality holds

$$\psi^\varepsilon \mathbf{1}_{\{|\psi|^\varepsilon \le \frac{\beta}{a(\varepsilon)}\}} = \psi \mathbf{1}_{\{|\psi| \le \frac{\beta}{a(\varepsilon)}\}},$$

which implies that ψ converges weakly to ψ, as $\varepsilon \to 0$. Thus from the statement (ii) of **Condition 3.1.1**,

$$\mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon}) \Rightarrow \mathcal{G}^0(\psi). \tag{D.7.11}$$

Using (D.7.2), statement (d) of **Lemma D.7.1** and

$$\frac{b(\varepsilon)}{\varepsilon} = \frac{1}{a^2(\varepsilon)}$$

we conclude

$$\begin{aligned}-b(\varepsilon) \ln \bar{\mathbb{E}}\Big[e^{-\frac{F(Z^\varepsilon)}{b(\varepsilon)}}\Big] &\le \frac{b(\varepsilon)}{\varepsilon}\mathfrak{L}_T(\varphi^\varepsilon) + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon}) \\ &\le \frac{1}{2}\int_{[0,T]\times\mathbb{R}^d} |\psi^\varepsilon|^2 d\nu_T + \kappa_3 \int_{[0,T]\times\mathbb{R}^d} a(\varepsilon)|\psi^\varepsilon|^3 d\nu_T + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon}) \\ &\le \frac{1}{2}(1 + 2\kappa_3\beta)\int_{[0,T]\times\mathbb{R}^d} |\psi|^2 d\nu_T + F \circ \mathcal{G}^\varepsilon(\varepsilon N^{\frac{1}{\varepsilon}\varphi^\varepsilon}).\end{aligned}$$

Sending $\varepsilon \to 0$, (D.7.11) implies that

$$\limsup_{\varepsilon\to 0} -b(\varepsilon) \ln \bar{\mathbb{E}}\Big[e^{-F(Z^\varepsilon)/b(\varepsilon)}\Big] \le \frac{1}{2}(1 + 2\kappa_3\beta)\int_{[0,T]\times\mathbb{R}^d} |\psi|^2 d\nu_T + F(\eta).$$

Sending $\beta \to 0$ in the last inequality, using (D.7.10) and (D.7.9) we conclude

$$\begin{aligned}\limsup_{\varepsilon\to 0} -b(\varepsilon) \ln \bar{\mathbb{E}}\Big[e^{-F(Z^\varepsilon)/b(\varepsilon)}\Big] &\le \frac{1}{2}\int_{[0,T]\times\mathbb{R}^d} |\psi|^2 d\nu_T + F(\eta) \\ &\le \mathbb{I}(\eta) + F(\eta) + \frac{\delta}{2} \\ &\le \inf_{\eta\in\mathbb{U}}[\mathbb{I}(\eta) + F(\eta)] + \delta.\end{aligned}$$

Since $\delta > 0$ is arbitrary the proof of (D.7.8) is complete.

□

Basic notation list

Numbers

- $\mathbb{R} = (-\infty, \infty)$ the set of real numbers.
- $\mathbb{R}^+ = [0, \infty)$ the set of the positive real numbers.
- $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.
- $\mathbb{R}^d = \{x = (x_1, \ldots, x_d) \mid x_i \in \mathbb{R},\ \text{for all } i \in \{1, \ldots, d\}\}$.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of the non-negative integers.
- $\mathbb{N}_1 = \mathbb{N} - \{0\}$
- $\mathbb{Z}$ the set of the integers.
- $\mathbb{Q}$ the set of the rational numbers.
- $i = \sqrt{-1}$ the imaginary unit.
- $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, for $a, b \in \mathbb{R}$.
- $a^+ = \max(a, 0)$, $a^- = -\min(a, 0)$, for $a \in \mathbb{R}$.
- $\lceil x \rceil = [x]$ the integer part of x, for $x \in \mathbb{R}$.
- $\ln x$ the logarithm of x in the natural base e, for $x > 0$.

Conventions and notation

- $a := b$ means a is defined to be b.
- Given a set $A \subset X$, $A^c = X - A$.
- $0 \times \infty = \frac{0}{0} = 0$.
- $\inf \emptyset = \infty$.

. $\langle x, y \rangle = x_1 y_1 + \dots x_d y_d$ and $|x| = \sqrt{x_1^2 + \dots x_d^2} = \sqrt{\langle x, x \rangle}$, for every $x, y \in \mathbb{R}^d$.

. A^T is the transpose of the matrix A.

. Γ is Euler's Gamma function.

. ν is a Lévy measure

. δ_x is a Dirac measure centered in x.

. Given a topological space X, $\mathrm{int}(A)$ stands for the topological interior of A and $\mathrm{cl}(A)$ stands for the topological closure of A.

. Given a metric space (M, d), $B_R(x)$ is the ball centered in x with radius R, $B_R = B_R(0)$, $B_R^c(x) = (B_R(x))^c$ and $d(x, F)$ is the distance of the point x to the closed set F.

. $\rightarrow$ stands for strong convergence, $\rightharpoonup$ for weak* convergence and $\Rightarrow$ for weak convergence of probability measures.

. $a_n \nearrow a$ if $(a_n)_{n \in \mathbb{N}}$ is increasing and converges to a.

. $a_n \searrow a$ if $(a_n)_{n \in \mathbb{N}}$ is decreasing and converges to a.

. $a_n = O(b_n)$ if there exists $M > 0$ such that $a_n \leq M b_n$, for all $n \in \mathbb{N}$.

. $a_n \simeq b_n$ if $\lim \frac{a_n}{b_n} \in \mathbb{R}$.

. $a_n \ll b_n$ if $\lim \frac{a_n}{b_n} = 0$.

. If μ, θ are measures we write $\mu \ll \theta$ if μ is absolutely continuous with respect to θ. $\mu \sim \theta$ is μ and θ are mutually absolutely continuous.

. Given a function $x : [0, T] \longrightarrow \mathbb{R}^d$, we write

$$\begin{aligned} x_t &= x(t) \\ x(t^-) &= \lim_{s \to t^-} x(s) \\ x(t^+) &= \lim_{s \to t^+} x(s) \\ \Delta_t x &= x(t) - x(t^-). \end{aligned}$$

- Given a function $F : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$, for every $(t,x) \in [0,T] \times \mathbb{R}^d$, we write

$$\begin{aligned}
F(t,x) &= F_t(x)\\
\partial_t F(t,x) &= \frac{\partial F}{\partial t}(t,x)\\
\partial_{x_i} F(t,x) &= \frac{\partial F}{\partial x_i}(t,x)\\
\nabla_x F(t,x) &= \Big(\partial_{x_1} F(t,x), \dots, \partial_{x_d} F(t,x)\Big)\\
\nabla_x^2 F(t,x) &= \Big(\frac{\partial F}{\partial x_i \partial x_j}(t,x)\Big)_{i,j=1,\dots,d}
\end{aligned}$$

Abbreviations

- ODE: ordinary differential equation.
- PDE: stochastic differential equation.
- PIDE: partial-integral differential equation.
- SDE: stochastic differential equation.
- BSDE: backward stochastic differential equation,
- FBSDE: forward-backward stochastic differential equation.
- LDP: large deviations principle.
- MDP: moderate deviations principle.
- PRM: Poisson random measure
- a.s.: almost surely
- a.e: almost everywhere.
- i.i.d.: independent and identically distributed.

Stochastic processes and distributions

- $Gaussian(\mu, \sigma)$ is a Gaussian distribution with mean μ and variance σ.
- $Poisson(\lambda)$ is a Poisson distribution with intensity $\lambda > 0$.
- $EXP(\lambda)$ is an Exponential distribution with intensity $\lambda > 0$.
- $(L_t)_{t \geq 0}$ is a Lévy process.

. $(B_t)_{t\geq 0}$ is a Brownian motion.

. $\tilde{N}^\theta$ is a compensated PRM with compensator $\theta ds \otimes \nu$, where ds is the Lebesgue measure defined on the Borel sets of $[0,\infty)$, ν a locally finite measure defined on the Borel sets of $\mathbb{R}^d$ and $\theta > 0$.

First exit times

. ε is the noise intensity parameter.

. $\nu(dz) = e^{-|z|^\alpha} dz$ where dz is the Lebesgue measure defined on $\mathbb{R}^d$ and $\alpha > 0$.

. X^ε is the exponentially light jump diffusion.

. $(\tilde{L}_t^\varepsilon)_{t\geq 0}$ is the stochastic perturbation, a compensated compound Poisson process with intensity $\frac{1}{\varepsilon} ds \otimes \nu$, $\varepsilon > 0$

. $D \subset \mathbb{R}^d$ is the pre-fixed domain.

. $\beta = \nu(\mathbb{R}^d) < \infty$ is the intensity.

. $\sigma^\varepsilon(x)$ is the first exit time of X^ε from D, $x \in D$.

. $\sigma_R^\varepsilon(x)$ is the first exit time of X^ε from a ball of radius $R > 0$, $x \in D$.

. τ_ρ^ε is the first exit time of $D - B_R(0)$, $x \in D$.

. $(W_i^\varepsilon)_{i\in\mathbb{N}}$ is the sequence of the jumps of X^ε.

. $(T_n^\varepsilon)_{n\in\mathbb{N}}$ is the sequence of the jumping times of X^ε.

. $(\tau_i^\varepsilon)_{i\in\mathbb{N}}$ is the sequence of the inter-jumping times of X^ε.

Functionals

. I : p. 17

. $\mathfrak{L}_T$: p. 23

. $R(\mathbb{P}||\mathbb{Q})$: p. 203

. $\mathfrak{L}_{t,T}$: p. 135

. $\tilde{\mathfrak{L}}_{t,T}$: p. 135

. $\bar{\mathfrak{L}}_{t,T} = \mathfrak{L}_{t,T} + \tilde{\mathfrak{L}}_{t,T}$: p. 136

. $\mathbb{J}$: p. 25

. $\mathbb{J}(\Phi)_{x,t}$: p. 26

. $V(x,z,t)$: p. 26

. $V(x,z)$: p. 26

. $\bar{V}$: p. 26

. $\tilde{\mathbb{I}}_0$: p. 28

. $\tilde{\mathbb{I}}_0(\Phi)_{x,t}$: p. 28

. $\tilde{\mathbb{I}}_1$: p. 29

. $\tilde{\mathbb{I}}_1(\Phi)_{x,t}$: p. 29

. $V_0(x,z,t)$: p. 28

. $V_0(x,z)$: p. 28

. $\bar{V}_0$: p. 28

. $V_1(x,z,t)$: p. 30

. $V_1(x,z)$: p. 30

. $\bar{V}_1$: p. 30

. $\mathbb{K}$: p. 36

. $\mathbb{L}$: p. 137

. $N^{\frac{1}{\varepsilon}\varphi_\varepsilon}$: p. 38

Spaces

. $(\mathcal{S},\mathcal{B}(\mathcal{S}))$ is a topological space with the Borel σ-algebra $\mathcal{B}(\mathcal{S})$.

. $\mathcal{P}(\mathcal{S})$ is the set of the probability measures on $(\mathcal{S},\mathcal{B}(\mathcal{S}))$.

. $M_b(\mathcal{S})$ is the space of the real-valued bounded measurable functions defined in $\mathcal{S}$ equipped with the sup norm $||.||_\infty$.

. $C_b(\mathcal{S})$ is the space of the real-valued continuous bounded functions defined in $\mathcal{S}$ equipped with the sup norm $||.||_\infty$.

. $C_c(\mathcal{S})$ is the space of the real-valued compactly supported continuous functions defined in $\mathcal{S}$ equipped with the sup norm $||.||_\infty$.

. $\mathbb{M}$: p. 19

. $\bar{\mathbb{M}}$: p. 20

. $\mathbb{M}_{t,T}$: p. 99

. $\bar{\mathbb{M}}_{t,T}$: p. 100

. $\bar{\mathcal{A}}^+$: p. 31

. $\bar{\mathcal{A}}^+_{b,n}$: p. 32

. $\bar{\mathcal{A}}^+_b$: p. 32

. $\bar{\mathcal{A}}^+_{t,T}$: p. 137

. $\bar{\mathcal{A}}^+_{t,T,b,n}$: p. 137

. $\bar{\mathcal{A}}^+_{t,T,b}$: p. 137

. S^M: p. 23

. $S^M_{+,\varepsilon}$: p. 70

. S^M_ε: p. 70

. $S^M_{t,T}$: p. 136

. $\mathcal{U}^M_+$: p. 32

. $\bar{\mathcal{U}}^M_{t,T}$: p. 137

. $\mathcal{U}^M_{+,\varepsilon}$: p. 71

. $\mathcal{U}^M_\varepsilon$: p. 71

. $L^2(\nu_T)$: p. 71

. $C_0([t,T],\mathbb{R}^d)$: p. 99

. $\mathbb{D}([0,T],\mathbb{R}^d)$: p. 25

. $\mathbb{V}$: p. 100

. $\bar{\mathbb{V}}$: p. 100

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